Error in constitutive relation for material identification

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1. **Concept of error in constitutive equation**

2. Time-harmonic elastodynamics

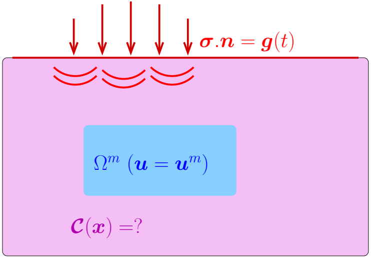
3. Transient elastodynamics

4. Transient or time-harmonic viscoelasticity

Identification of material constitutive properties

Generic problem: Identify (possibly heterogeneous) material parameters from overdetermined data, e.g.:

- kinematic data on the boundary;
- vibrational data (eigenfrequencies, eigenmodes at sensors);
- full-field kinematic response of solid under dynamical excitation...



Identification often based on minimizing a data misfit functional

- (weighted) least squares...;
- Reciprocity residuals (reciprocity gap method, virtual fields method);
- Bayesian approaches;
- Error in constitutive equation (ECR)

Concept of error in constitutive equation (ECR)

- Simplest version of ECR (small-strain linear elasticity):

$$\mathcal{E}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) := \frac{1}{2}\int_{\Omega}(\boldsymbol{\sigma}-\boldsymbol{\mathcal{C}}\!:\!\boldsymbol{\varepsilon}[\boldsymbol{u}])\!:\!\boldsymbol{\mathcal{C}}^{-1}\!:\!(\boldsymbol{\sigma}-\boldsymbol{\mathcal{C}}\!:\!\boldsymbol{\varepsilon}[\boldsymbol{u}])\,\mathrm{d}V$$
$$\widetilde{\mathcal{E}}(\boldsymbol{\mathcal{C}}) := \min_{\boldsymbol{u}\in\mathsf{KA},\boldsymbol{\sigma}\in\mathsf{SA}} \mathcal{E}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}})$$

- $\widetilde{\mathcal{E}}(\boldsymbol{\mathcal{C}})$: Energy-based measure of mismatch between KA and SA spaces for given domain, material and loading $\Longrightarrow$ mechanically meaningful cost functional.
  - ▷ First introduced for error estimation in FEM [Ladevèze, Leguillon 1983];
  - ▷ Soon also proved useful for identification problems [e.g. Reynier 1990]
  - ▷ Similar ideas independently in EIT [Kohn, Vogelius, McKenney, c. 1990]
  - ▷ Also useful for Cauchy / data completion problems [Andrieux, Ben Abda 2006]
  - ▷ Plasticity, damage... [e.g. Latourte et al. 2007, Marchand et al. 2018]
  - ▷ Special case of Fenchel error $\Longrightarrow$ ECR for generalized standard materials
- Time-harmonic formulation for e.g. FE model updating
  [e.g. Reynier 90; Moine 97; Deraemaeker 01; Banerjee et al. 13; Aquino, B 19; **many** more]
- Time domain formulation
  [Allix, Feissel, Nguyen 05; Allix, Feissel 06] (spatially 1D), [Aquino, B 14]

Error in constitutive relation (ECR)

$\boxed{\mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{C}})=0}$ $\iff$ (elastic) constitutive eq. satisfied (in $\boldsymbol{L}^2(\Omega)$)

- Hence, elastic equilibrium problem, e.g.

$$\begin{aligned}
\boldsymbol{v} &= \overline{\boldsymbol{u}} \quad \text{on } S_u && \text{(compatibility)}\\
-\text{div}\,\boldsymbol{\tau} &= \boldsymbol{f} \quad \text{in } \Omega, \qquad \boldsymbol{\tau}\cdot\boldsymbol{n} = \bar{\boldsymbol{t}} \quad \text{on } S_\mathsf{T} && \text{(equilibrium)}\\
\boldsymbol{\tau} &= \boldsymbol{\mathcal{C}}:\boldsymbol{\varepsilon}[\boldsymbol{v}] \quad \text{in } \Omega && \text{(constitutive)}
\end{aligned}$$

  as ECR minimization (for given material):

$$(\boldsymbol{u},\boldsymbol{\sigma}) = \underset{(\boldsymbol{v},\boldsymbol{\tau})\in\mathcal{C}(\bar{\boldsymbol{u}})\times\mathcal{S}(\bar{\boldsymbol{t}},\,\boldsymbol{f})}{\arg\min}\ \mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{C}})$$

  Typically: $\mathsf{KA} = \big\{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega),\ \boldsymbol{v}$ satisfies (compatibility)$\big\}$
  $\mathsf{SA} = \big\{\boldsymbol{\tau}\in\boldsymbol{H}_{\mathsf{div}}(\Omega),\ \boldsymbol{\tau}=\boldsymbol{\tau}^{\mathsf{s}}$ and satisfies (equilibrium)$\big\}$

- Combines potential and complementary energy minimizations:

$$\boxed{\mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{C}}) = \mathcal{P}(\boldsymbol{v},\boldsymbol{\mathcal{C}}) + \mathcal{P}^\star(\boldsymbol{\tau},\boldsymbol{\mathcal{C}})}$$

$$\mathcal{P}(\boldsymbol{v}) = \frac{1}{2}\int_\Omega \boldsymbol{\varepsilon}[\boldsymbol{v}]:\boldsymbol{\mathcal{C}}:\boldsymbol{\varepsilon}[\boldsymbol{v}]\ \mathrm{d}V - \int_\Omega \rho\boldsymbol{f}.\boldsymbol{v}\ \mathrm{d}V - \int_{S_T}\bar{\boldsymbol{t}}.\boldsymbol{v}\,\mathrm{d}S$$

$$\mathcal{P}^\star(\boldsymbol{\tau}) = \frac{1}{2}\int_\Omega \boldsymbol{\tau}:\boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\tau}\ \mathrm{d}V - \int_{S_u}[\boldsymbol{\tau}.\boldsymbol{n}].\bar{\boldsymbol{u}}\,\mathrm{d}S_x$$

Variational formulations, error estimation

This retrieves well-known uncoupled minimizations of potential and complementary energies:

$$\boldsymbol{u} = \arg\min_{\boldsymbol{v}\in\mathcal{C}(\bar{\boldsymbol{u}})} \mathcal{P}(\boldsymbol{v},\boldsymbol{\mathcal{C}}) \qquad \boldsymbol{\sigma} = \arg\min_{\boldsymbol{\tau}\in\mathcal{S}(\bar{\boldsymbol{t}},\,\boldsymbol{f})} \mathcal{P}^{\star}(\boldsymbol{\tau},\boldsymbol{\mathcal{C}})$$

(i) $\mathcal{E}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) = \mathcal{E}(\boldsymbol{\mathcal{C}}) = 0$ indicates that $\boldsymbol{\mathcal{C}}$ is consistent with kinematic and static data. Performing **either** minimization suffices.

- E.g. $\boldsymbol{u} = \arg\min_{\boldsymbol{v}\in\mathcal{C}(\bar{\boldsymbol{u}})} \mathcal{P}(\boldsymbol{v},\boldsymbol{\mathcal{C}})$ then $\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}}:\boldsymbol{\varepsilon}[\boldsymbol{u}]$

(ii) $\mathcal{E}_h(\boldsymbol{u}_h,\boldsymbol{\sigma}_h,\boldsymbol{\mathcal{C}}) = E_h(\boldsymbol{\mathcal{C}}) \geq 0$ allows to defini error indicators *additive w.r.t. finite elements*

- Original motivation for introducing the ECR concept

(iii) $\mathcal{E}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) = \mathcal{E}(\boldsymbol{\mathcal{C}}) > 0$ indicates that $\boldsymbol{\mathcal{C}}$ is **not** consistent with kinematic and static data.

- E.g. material identification / imaging problem with incorrectly known constitutive properties.

Generalization to other constitutive models

- (elastic) ECR given in terms of free energy and conjugate potentials $\psi, \psi^\star$

$$\mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{C}}) = \int_\Omega \big(\underbrace{\psi(\boldsymbol{\varepsilon}[\boldsymbol{v}]) + \psi^\star(\boldsymbol{\tau}) - \boldsymbol{\tau}:\boldsymbol{\varepsilon}[\boldsymbol{v}]}_{\text{Legendre-Fenchel gap, } \geq 0} \big) \, \mathrm{d}V$$

$$\psi(\boldsymbol{\varepsilon}) := \tfrac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\mathcal{C}}:\boldsymbol{\varepsilon}, \quad \psi^\star(\boldsymbol{\tau}) := \tfrac{1}{2}\boldsymbol{\tau}:\boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\tau}.$$

- More generally (small-strain nonlinear elasticity), same definition of $\mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{C}})$ if
  - ▷ ψ: convex free energy density with $\psi \geq 0$ and $\psi(\mathbf{0}) = 0$;
  - ▷ $\psi^\star(\boldsymbol{\tau})$: (convex) conjugate potential

$$\boxed{\psi^\star(\boldsymbol{\tau}) := \sup_{\boldsymbol{\varepsilon}} \big(\boldsymbol{\tau}:\boldsymbol{\varepsilon} - \psi(\boldsymbol{\varepsilon}) \big)}$$

- Further generalization: generalized standard materials (GSMs) [Halphen, Nguyen 75; Germain, Nguyen, Suquet 83], in terms of free energy ψ and dissipation φ potentials:
$$\boldsymbol{\tau} = \boldsymbol{\tau}^{\text{rev}} + \boldsymbol{\tau}^{\text{irr}} = \partial_\varepsilon \psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) + \partial_{\dot{\varepsilon}}\varphi(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}}), \qquad \boldsymbol{A} = -\partial_\alpha \psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) = \partial_{\dot{\alpha}}\varphi(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}})$$
($\boldsymbol{\alpha}$: internal variables, $\boldsymbol{A}$: conjugate thermodynamic forces). ECR functionals then defined in terms of Legendre-Fenchel gaps

$$\begin{aligned} \psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) + \psi^\star(\boldsymbol{\tau}^{\text{rev}},\boldsymbol{A}) - \boldsymbol{\tau}^{\text{rev}}:\boldsymbol{\varepsilon} + \boldsymbol{A}:\boldsymbol{\alpha} \geq 0, \\ \varphi(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}}) + \varphi^\star(\boldsymbol{\tau}^{\text{irr}},\boldsymbol{A}) - \boldsymbol{\tau}^{\text{irr}}:\dot{\boldsymbol{\varepsilon}} - \boldsymbol{A}:\dot{\boldsymbol{\alpha}} \geq 0. \end{aligned}$$

ECR functionals may be defined (using Legendre-Fenchel gaps) for all GSMs.

ECR-based material identification

- Minimization of pure ECR:

$$\mathcal{C} = \arg\min_{\mathcal{B}\in\mathcal{Q}} \Big\{ \min_{\boldsymbol{v}\in\mathsf{KA},\boldsymbol{\tau}\in\mathsf{SA}} \mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\mathcal{B}) \Big\}$$

$$\mathsf{KA} = \big\{ \boldsymbol{v}\in\boldsymbol{H}^1(\Omega),\ \boldsymbol{v} \text{ verifies all kinematic constraints and data} \big\}$$

$$\mathsf{SA} = \big\{ \boldsymbol{\tau}\in\boldsymbol{H}_{\mathsf{div}}(\Omega),\ \boldsymbol{\tau}=\boldsymbol{\tau}^{\mathsf{s}} \text{ and verifies all balance constraints}$$

However, exact imposition of noisy data usually inadvisable.

- Minimization of **modified ECR** (MECR):

$$\Lambda_\kappa(\boldsymbol{v},\boldsymbol{\tau},\mathcal{C}) := \mathcal{E}(\boldsymbol{v},\boldsymbol{\tau},\mathcal{C}) + \kappa\mathcal{D}(\boldsymbol{u}-\boldsymbol{u}_{\mathsf{obs}})$$

$\mathcal{E}$: original ECR, $\quad\mathcal{D}$: quadratic ≥ 0, e.g. $\quad \mathcal{D}(\boldsymbol{w}) = \dfrac{1}{2}\displaystyle\int_{\Omega^m} |\boldsymbol{w}|^2 \,\mathrm{d}V.$

  - ▷ Enforces kinematic data via penalization (so data not embedded in KA space)
  - ▷ κ: tunable penalty (or coupling) parameter, akin to regularization (see later)

- **Reduced MECR** functional:

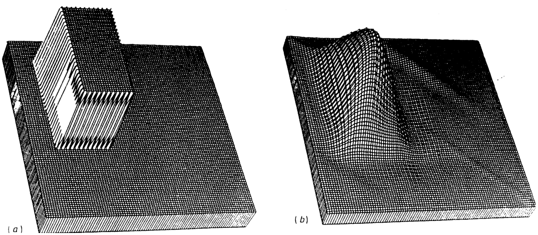
$$(\boldsymbol{u},\boldsymbol{\sigma}) := \arg\min_{\boldsymbol{v}\in\mathsf{KA},\boldsymbol{\tau}\in\mathsf{SA}} \Lambda_\kappa(\boldsymbol{v},\boldsymbol{\tau},\mathcal{C}), \qquad \tilde{\Lambda}_\kappa(\mathcal{C}) := \Lambda_\kappa(\boldsymbol{u},\boldsymbol{\sigma},\mathcal{C}) \tag{PM}$$

  - ▷ Quadratic partial minimization problem (see later), i.e. linear stationarity eqs.
  - ▷ $\boldsymbol{u}=\boldsymbol{u}[\mathcal{C}], \boldsymbol{\sigma}=\boldsymbol{\sigma}[\mathcal{C}]$ best compromise between (i) constitutive guess $\mathcal{C}$, (ii) measurements
  - ▷ $\tilde{\Lambda}_\kappa(\mathcal{C}) \neq 0$: residual MECR value reflecting incorrectly-known material.

ECR-type functional for electrical impedance imaging

- Equations (v: potential, $\boldsymbol{e}$: electric field, $\boldsymbol{q}$: current):
  $$\operatorname{div} \boldsymbol{q}(\boldsymbol{x}) = 0, \qquad \boldsymbol{q}(\boldsymbol{x}) = a(\boldsymbol{x})\boldsymbol{e}(\boldsymbol{x}), \qquad \boldsymbol{e}(\boldsymbol{x}) = -\boldsymbol{\nabla} v(\boldsymbol{x})$$
- ECR-type functional for N experiments with data $\bar{v}$ for v and $\bar{q}$ for $\boldsymbol{q}.\boldsymbol{n}$ on $\partial\Omega$:
  $$\boxed{\mathcal{E}(a, v_1, \ldots, v_N, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N) = \sum_{i=1}^{N} \int_\Omega \|a^{1/2}\boldsymbol{\nabla} v + a^{-1/2}\boldsymbol{q}\|^2 \, \mathrm{d}V}$$

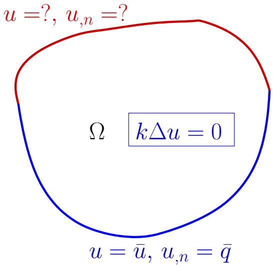
Note:
$$\|a^{1/2}\boldsymbol{\nabla} v + a^{-1/2}\boldsymbol{q}\|^2 = \frac{1}{a}|\boldsymbol{q} + a\boldsymbol{\nabla} v\|^2 = \frac{1}{a}|\boldsymbol{q} - a\boldsymbol{e}\|^2$$



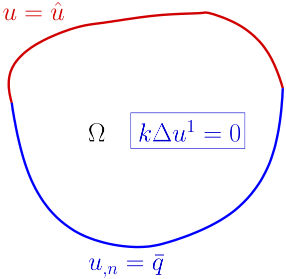
(a) "true" σ; reconstructions (b) no data noise.

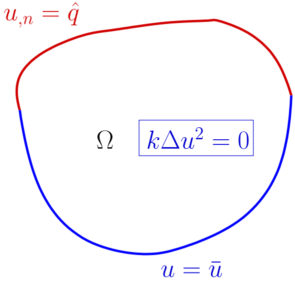
Kohn R. V., Vogelius M., *Comm. Pure Appl. Math.* **40**:745–777 (1987)
Kohn R. V., McKenney A., *Inverse Problems* **6**:389–414 (1990)

Energy (ECR-like) functional for ill-posed boundary value problems



$$\boxed{\mathcal{E}(\hat{u},\hat{q}) = \int_\Omega k\boldsymbol{\nabla}(u^1 - u^2)\cdot\boldsymbol{\nabla}(u^1 - u^2)\ \mathsf{d}V}$$

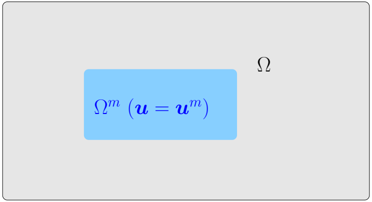




Andrieux S., Baranger T., Ben Abda A., *Inverse Problems* **22**:115–133 (2006)

Framework

- Elastodynamic ECR-based reconstruction of heterogeneous elastic properties
- No prescribed boundary data:
  - ▷ Well-posed forward problem *a priori* unclear (in contrast to usual inversion sitiations);
- This talk (based on [Aquino, B; SIAP (2019)]): internal kinematical data only
  - ▷ (possibly overdetermined) boundary measurements may also be accounted for
  - ▷ Cases with well-posed BCs covered as special cases



$\partial\Omega = \Gamma$ (BC unknown)

Balance (SA): $\big(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}[\boldsymbol{w}]\big)_\Omega - \omega^2\big(\rho\boldsymbol{u}, \boldsymbol{w}\big)_\Omega = \mathcal{F}(\boldsymbol{w})$ for all $\boldsymbol{w} \in \mathcal{W} := \boldsymbol{H}^1_0(\Omega)$,

Kinematic compatibility (KA): $\boldsymbol{u} \in \mathcal{U} := \boldsymbol{H}^1(\Omega), \quad \boldsymbol{\varepsilon}[\boldsymbol{u}] = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{T}})$ in Ω,

Constitutive (linear elastic): $\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}[\boldsymbol{u}]$ in Ω.

Modified ECR (MECR) functional

MECR functional:

$$\Lambda_\kappa(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) := \mathcal{E}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) + \kappa\mathcal{D}(\boldsymbol{u}-\boldsymbol{u}_{\text{obs}})$$

$$\mathcal{E}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) := \frac{1}{2}\int_\Omega (\boldsymbol{\sigma}-\boldsymbol{\mathcal{C}}:\boldsymbol{\varepsilon}[\boldsymbol{u}]):\boldsymbol{\mathcal{C}}^{-1}:(\boldsymbol{\sigma}-\boldsymbol{\mathcal{C}}:\boldsymbol{\varepsilon}[\boldsymbol{u}])\,\mathrm{d}V$$

$$\mathcal{D}:\ \text{quadratic} > 0, \text{ e.g. } \ \mathcal{D}(\boldsymbol{w}) = \frac{1}{2}\int_{\Omega^m} |\boldsymbol{w}|^2\,\mathrm{d}V$$

- ▷ Enforces kinematic data via penalization (so data not embedded in KA space)
- ▷ κ: tunable penalty (or coupling) parameter, akin to regularization (see later)

Reduced MECR:

$$(\boldsymbol{u},\boldsymbol{\sigma}) := \arg\min_{\boldsymbol{v}\in\text{KA},\boldsymbol{\tau}\in\text{SA}} \Lambda_\kappa(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{C}}), \qquad \tilde{\Lambda}_\kappa(\boldsymbol{\mathcal{C}}) := \Lambda_\kappa(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) \tag{PM}$$

- ▷ Quadratic minimization problem (see later), i.e. linear stationarity eqs.
- ▷ $\boldsymbol{u},\boldsymbol{\sigma}$ best compromise between (i) constitutive guess $\boldsymbol{\mathcal{C}}$, (ii) measurements
- ▷ $\tilde{\Lambda}_\kappa(\boldsymbol{\mathcal{C}}) \neq 0$: residual MECR value reflecting incorrectly-known material.

Constitutive identification problem

Full-space approach:

$$(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) := \operatorname*{arg\,min}_{\boldsymbol{v}\in\mathcal{U},\,\boldsymbol{\tau}\in\mathcal{S}(\boldsymbol{v}),\,\boldsymbol{\mathcal{C}}\in\mathcal{Q}} \Lambda_\kappa(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{C}}).$$

Reduced-space approach: based on the reduced MECR:

$$(\boldsymbol{u},\boldsymbol{\sigma}) := \operatorname*{arg\,min}_{\boldsymbol{v}\in\mathcal{U},\,\boldsymbol{\tau}\in\mathcal{S}(\boldsymbol{v})} \Lambda_\kappa(\boldsymbol{v},\boldsymbol{\tau},\boldsymbol{\mathcal{C}}), \qquad \tilde{\Lambda}_\kappa(\boldsymbol{\mathcal{C}}) := \Lambda_\kappa(\boldsymbol{u}[\boldsymbol{\mathcal{C}}],\boldsymbol{\sigma}[\boldsymbol{\mathcal{C}}],\boldsymbol{\mathcal{C}}) \tag{PM}$$

(at least) two approaches for the constitutive identification problem:

1. **Minimize $\tilde{\Lambda}_\kappa(\boldsymbol{\mathcal{C}})$ (e.g. using CG, BFGS...)**

$$\boldsymbol{\mathcal{C}}^\star := \operatorname*{arg\,min}_{\boldsymbol{\mathcal{C}}\in\mathcal{Q}} \tilde{\Lambda}_\kappa(\boldsymbol{\mathcal{C}})$$

   Each evaluation of $\tilde{\Lambda}_\kappa(\boldsymbol{\mathcal{C}}), \tilde{\Lambda}'_\kappa(\boldsymbol{\mathcal{C}})$ needs to **solve (PM)**.

2. **Minimize $\Lambda_\kappa(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}})$ via alternate directions**
   - Field update (global) **via (PM)**, $\boldsymbol{\mathcal{C}}$ fixed: $\boldsymbol{u},\boldsymbol{\sigma}$
   - Constitutive update (local, often closed-form)

$$\boldsymbol{\mathcal{C}}^\star := \operatorname*{arg\,min}_{\boldsymbol{\mathcal{C}}\in\mathcal{Q}} \Lambda_\kappa(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}})$$

Problem (PM) plays a key role.

Stationarity equations

Lagrangian (incorporating interior dynamical balance constraint with multiplier $\boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega)$):

$$\mathcal{L}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{w},\boldsymbol{\mathcal{C}}) := \mathcal{E}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) + \kappa\mathcal{D}(\boldsymbol{u}-\boldsymbol{u}_{\text{obs}}) + \Big\{ \big(\boldsymbol{\sigma},\boldsymbol{\varepsilon}[\boldsymbol{w}]\big)_\Omega - \omega^2\big(\rho\boldsymbol{u},\boldsymbol{w}\big)_\Omega - \mathcal{F}(\boldsymbol{w}) \Big\}, \qquad \boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega)$$

First-order optimality conditions:

$$\partial_{\boldsymbol{u}}\mathcal{L} = 0, \quad \partial_{\boldsymbol{\sigma}}\mathcal{L} = 0, \quad \partial_{\boldsymbol{w}}\mathcal{L} = 0, \quad \partial_{\mathcal{C}}\mathcal{L} = 0 \; \boxed{\partial_{\boldsymbol{u}}\mathcal{L} = 0, \quad \partial_{\boldsymbol{\sigma}}\mathcal{L} = 0, \quad \partial_{\boldsymbol{w}}\mathcal{L} = 0,} \quad \partial_{\mathcal{C}}\mathcal{L} =$$

(i) Partial minimization of $(\boldsymbol{u},\boldsymbol{\sigma}) \mapsto \Lambda_\kappa(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) := \mathcal{E}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) + \kappa\mathcal{D}(\boldsymbol{u}-\boldsymbol{u}_{\textbf{obs}})$:

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}}:\boldsymbol{\varepsilon}[\boldsymbol{u}-\boldsymbol{w}], \qquad \begin{aligned} &\mathcal{A}(\boldsymbol{w},\widetilde{\boldsymbol{w}},\boldsymbol{\mathcal{C}}) + \mathcal{B}(\boldsymbol{u},\widetilde{\boldsymbol{w}},\boldsymbol{\mathcal{C}}) = \mathcal{F}(\widetilde{\boldsymbol{w}}) && \text{for all } \widetilde{\boldsymbol{w}} \in \mathcal{W} \\ &\mathcal{B}(\widetilde{\boldsymbol{u}},\boldsymbol{w},\boldsymbol{\mathcal{C}}) - \kappa\mathcal{D}(\boldsymbol{u},\widetilde{\boldsymbol{u}}) = -\kappa\mathcal{D}(\boldsymbol{u}_{\text{obs}},\widetilde{\boldsymbol{u}}) && \text{for all } \widetilde{\boldsymbol{u}} \in \mathcal{U} \end{aligned}$$

$\mathcal{B}(\cdot,\cdot,\boldsymbol{\mathcal{C}}) := \mathcal{A}(\cdot,\cdot,\boldsymbol{\mathcal{C}}) - \omega^2\mathcal{M}(\cdot,\cdot)$: dynamical stiffness bilinear form

(i) Partial minimization of $(\boldsymbol{u},\boldsymbol{\sigma}) \mapsto \Lambda_\kappa(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) := \mathcal{E}(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\mathcal{C}}) + \kappa\mathcal{D}(\boldsymbol{u}-\boldsymbol{u}_{\textbf{obs}})$:

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}}:\boldsymbol{\varepsilon}[\boldsymbol{u}-\boldsymbol{w}], \qquad \begin{aligned} &\mathcal{A}(\boldsymbol{w},\widetilde{\boldsymbol{w}},\boldsymbol{\mathcal{C}}) + \mathcal{B}(\boldsymbol{u},\widetilde{\boldsymbol{w}},\boldsymbol{\mathcal{C}}) = \mathcal{F}(\widetilde{\boldsymbol{w}}) && \text{for all } \widetilde{\boldsymbol{w}} \in \mathcal{W} \\ &\mathcal{B}(\widetilde{\boldsymbol{u}},\boldsymbol{w},\boldsymbol{\mathcal{C}}) - \kappa\mathcal{D}(\boldsymbol{u},\widetilde{\boldsymbol{u}}) = -\kappa\mathcal{D}(\boldsymbol{u}_{\text{obs}},\widetilde{\boldsymbol{u}}) && \text{for all } \widetilde{\boldsymbol{u}} \in \mathcal{U} \end{aligned}$$

$\mathcal{B}(\cdot,\cdot,\boldsymbol{\mathcal{C}}) := \mathcal{A}(\cdot,\cdot,\boldsymbol{\mathcal{C}}) - \omega^2\mathcal{M}(\cdot,\cdot)$: dynamical stiffness bilinear form

(ii) Nonlinear stationarity equation on $\boldsymbol{\mathcal{C}}$:

Well-posedness of stationarity problem (finite-dimensional case)

- Coupled stationarity problem (replaces forward + adjoint):

$$\begin{aligned} \mathcal{A}(\boldsymbol{w},\widetilde{\boldsymbol{w}},\boldsymbol{\mathcal{C}})+\mathcal{B}(\boldsymbol{u},\widetilde{\boldsymbol{w}},\boldsymbol{\mathcal{C}}) &= \mathcal{F}(\widetilde{\boldsymbol{w}}) && \text{for all } \widetilde{\boldsymbol{w}}\in\mathcal{W} \\ \mathcal{B}(\widetilde{\boldsymbol{u}},\boldsymbol{w},\boldsymbol{\mathcal{C}})-\kappa\mathcal{D}(\boldsymbol{u},\widetilde{\boldsymbol{u}}) &= -\kappa\mathcal{D}(\boldsymbol{u}_{\text{obs}},\widetilde{\boldsymbol{u}}) && \text{for all } \widetilde{\boldsymbol{u}}\in\mathcal{U} \end{aligned}$$

- Let $\dim(\mathcal{U})=n$, $\dim(\mathcal{W})=m\leq n$.

$$\begin{aligned} \mathcal{A}:\mathcal{W}\times\mathcal{W}\to\mathbb{R} &\quad\longrightarrow \boldsymbol{A}\in\mathbb{R}^{m\times m} \\ \mathcal{B}=\mathcal{A}-\omega^2\mathcal{M}:\mathcal{U}\times\mathcal{W}\to\mathbb{R} &\quad\longrightarrow \boldsymbol{B}\in\mathbb{R}^{m\times n} \\ \mathcal{D}:\mathcal{U}\times\mathcal{U}\to\mathbb{R} &\quad\longrightarrow \boldsymbol{D}\in\mathbb{R}^{n\times n} \end{aligned}$$

- Discretized stationarity problem (BC setup such that $\boldsymbol{A}$ invertible):

$$\boxed{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^{\mathsf{T}} & -\kappa\boldsymbol{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{w} \\ \boldsymbol{u} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F} \\ \kappa\boldsymbol{D}\boldsymbol{u}^m \end{Bmatrix}}$$

- For any $(\boldsymbol{u},\boldsymbol{w})$ solving the homogeneous system:

$$\boldsymbol{u}^{\mathsf{T}}\boldsymbol{B}^{\mathsf{T}}\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{u}+\kappa\boldsymbol{u}^{\mathsf{T}}\boldsymbol{D}\boldsymbol{u}=0$$

  Discrete stationarity system therefore well-posed if $\boxed{N(\boldsymbol{B})\cap N(\boldsymbol{D})=\{\boldsymbol{0}\}}$

- Interpretation: available data must (more than) compensate lack of information on BCs

Well-posedness of stationarity problem (continuous case)

- Let $\mathcal{H} := \{ \boldsymbol{u} \in \mathcal{U}, \mid \mathcal{B}(\boldsymbol{u}, \widetilde{\boldsymbol{w}}, \boldsymbol{\mathcal{C}}) = 0 \text{ for all } \widetilde{\boldsymbol{w}} \in \mathcal{W} \}$ underdetermined BCs on $\boldsymbol{u}$ (recall $\mathcal{W} \subset \mathcal{U}$)

Theorem (W. Aquino, MB, 2019):

Assume $\mathcal{D}$ coercive on $\mathcal{H} \times \mathcal{H}$ (i.e. data compensates insufficient BC information).
The two-field stationarity problem

$$\begin{aligned}
\mathcal{A}(\boldsymbol{w}, \widetilde{\boldsymbol{w}}, \boldsymbol{\mathcal{C}}) + \mathcal{B}(\boldsymbol{u}, \widetilde{\boldsymbol{w}}, \boldsymbol{\mathcal{C}}) &= \mathcal{F}(\widetilde{\boldsymbol{w}}) && \text{for all } \widetilde{\boldsymbol{w}} \in \mathcal{W} \\
\mathcal{B}(\widetilde{\boldsymbol{u}}, \boldsymbol{w}, \boldsymbol{\mathcal{C}}) - \kappa \mathcal{D}(\boldsymbol{u}, \widetilde{\boldsymbol{u}}) &= -\kappa \mathcal{D}(\boldsymbol{u}_{\text{obs}}, \widetilde{\boldsymbol{u}}) && \text{for all } \widetilde{\boldsymbol{u}} \in \mathcal{U}
\end{aligned}$$

has a unique solution $(\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{U} \times \mathcal{W}$, which is continuous in $\mathcal{F}, \boldsymbol{u}_{\text{obs}}$

Proof method: Treat stationarity pb. as perturbed mixed problem [Boffi, Brezzi, Fortin 13].

Highlights: Stationarity problem is well-posed if sufficient full-field data available

- holds for **all frequencies**
- holds for (almost) all cases of BCs **(including underdetermined BCs)**
- includes well-posed BC case, for which $\mathcal{U} = \mathcal{W} = \{ \boldsymbol{u} \in \boldsymbol{H}^1(\Omega),\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_{\text{D}} \}$ (say)

Shortcoming: Assumed coercivity of $\mathcal{D}$ on $\mathcal{H} \times \mathcal{H}$ in H^1-norm.

Comparison with minimization of measurement misfit

- Conventional PDE-constrained constitutive identification: relation $\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}[\boldsymbol{u}]$ enforced.
- Lagrangian:

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{\mathcal{C}}) := \mathcal{D}(\boldsymbol{u} - \boldsymbol{u}_{\text{obs}}) + \Big\{ \big(\boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}[\boldsymbol{u}], \boldsymbol{\varepsilon}[\boldsymbol{w}] \big)_{\Omega} - \omega^2 \big(\rho \boldsymbol{u}, \boldsymbol{w} \big)_{\Omega} - \mathcal{F}(\boldsymbol{w}) \Big\}, \quad \boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega)$$

- First-order optimality conditions:

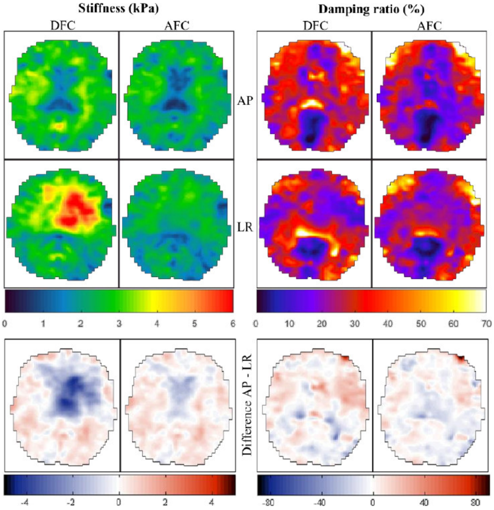
$$\partial_{\boldsymbol{u}} \mathcal{L} = 0, \qquad \partial_{\boldsymbol{w}} \mathcal{L} = 0, \qquad \partial_{\mathcal{C}} \mathcal{L} = 0$$

Forward and adjoint problems (coupled if $\mathcal{W} \neq \mathcal{U}$):

$$\begin{aligned} \mathcal{B}(\boldsymbol{u}, \widetilde{\boldsymbol{w}}, \boldsymbol{\mathcal{C}}) &= \mathcal{F}(\widetilde{\boldsymbol{w}}) && \text{for all } \widetilde{\boldsymbol{w}} \in \mathcal{W} \\ \mathcal{B}(\widetilde{\boldsymbol{u}}, \boldsymbol{w}, \boldsymbol{\mathcal{C}}) - \mathcal{D}(\boldsymbol{u}, \widetilde{\boldsymbol{u}}) &= -\mathcal{D}(\boldsymbol{u}_{\text{obs}}, \widetilde{\boldsymbol{u}}) && \text{for all } \widetilde{\boldsymbol{u}} \in \mathcal{U} \end{aligned}$$

Magnetic resonance elastography (S. Kurtz et al., Montpellier U. / Sherbrooke U.)

Coupled forward-adjoint implemented (for freq. domain elastodynamic sensing using 3D internal kinematic data) within a multizone approach



S. Kurtz, PhD 2023

Hessian of reduced MECR functional

- Consider reduced MECR functional $\tilde{\Lambda}_\kappa(\boldsymbol{\mathcal{C}}) := \Lambda_\kappa(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\mathcal{C}})$
- Expression of Hessian $\tilde{\Lambda}''_\kappa(\boldsymbol{\mathcal{C}})$ established in terms of $\boldsymbol{u}_{\mathcal{C}}, \boldsymbol{w}_{\mathcal{C}}$ and $\boldsymbol{u}'_{\mathcal{C}}, \boldsymbol{w}'_{\mathcal{C}}$ (not shown)
- Large-κ expansion of stationarity solution (suitable if data noise low enough):

$$(\boldsymbol{u}, \boldsymbol{w}_{\mathcal{C}}) = (\boldsymbol{u}_0, \boldsymbol{w}_0) + \kappa^{-1}(\boldsymbol{u}_1, \boldsymbol{w}_1) + \dots \qquad \text{(E)}$$

  with $(\boldsymbol{u}_\ell, \boldsymbol{w}_\ell)$ $(\ell = 0, 1, 2, \dots)$ defined as solutions of two-field problems.
- Insert (E) in $\tilde{\Lambda}''_\kappa(\boldsymbol{\mathcal{C}})$ gives

Theorem (MB, W. Aquino, 2019)

For any $\hat{\boldsymbol{\mathcal{C}}}$ such that supp$(\hat{\boldsymbol{\mathcal{C}}}) \subset \Omega^m$:

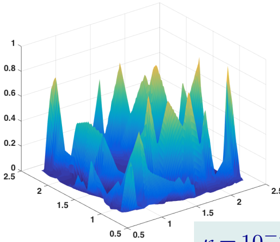
$$\boxed{\tilde{\Lambda}''_\kappa(\boldsymbol{\mathcal{C}})[\hat{\boldsymbol{\mathcal{C}}}] = \tilde{\Lambda}''_0(\boldsymbol{\mathcal{C}})[\hat{\boldsymbol{\mathcal{C}}}] + \kappa^{-1}\tilde{\Lambda}''_1(\boldsymbol{\mathcal{C}})[\hat{\boldsymbol{\mathcal{C}}}] + o(\kappa^{-1})}$$

where $\tilde{\Lambda}''_0(\boldsymbol{\mathcal{C}})[\hat{\boldsymbol{\mathcal{C}}}] \geq 0$, $\tilde{\Lambda}''_1(\boldsymbol{\mathcal{C}})[\hat{\boldsymbol{\mathcal{C}}}]$ sign-indefinite

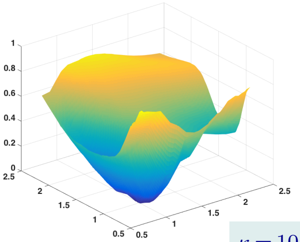
- $\tilde{\Lambda}''_\kappa$ is positive, i.e. $\tilde{\Lambda}_\kappa$ is "asymptotically convex", in the $\kappa \to \infty$ limit if supp$(\hat{\boldsymbol{\mathcal{C}}}) \subset \Omega^m$
- No such result available for L^2 minimization, even with complete internal data

Hessian of MECR functional

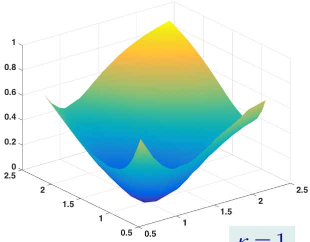
$\Omega=(0,1),\quad -(Eu')'-(2\pi f)^2u=b,\qquad u(0)=u(1)=0,\qquad E=\chi_{[0,1/2]}E_1+\chi_{[1/2,1]}E_2$



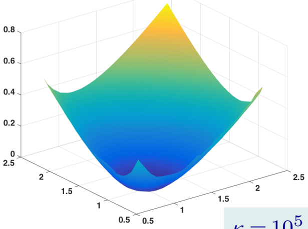
$\kappa=10^{-3}$



$\kappa=10^{-1}$



$\kappa=1$



$\kappa=10^{5}$

MECR-based alternate-direction reconstruction algorithm

$\boxed{\partial_{\mathcal{C}}\mathcal{L} = \mathbf{0}} \implies$ updating equations for the moduli.

- Assume unknown bulk and shear moduli B_E, G_E over element (groupings) E.
- Alternating direction minimization:
  - ▷ At iteration q, available estimates G^{q-1} and B^{q-1} of moduli.
  - ▷ Obtain $\boldsymbol{u}^q$ and $\boldsymbol{w}^q$ by solving the coupled systems of equations
  - ▷ Update the moduli in each element or subdomain as

$$B^q = \frac{\|s_k^q\|_{E,2}}{\|e_k^q\|_{E,2}}, \qquad 2G^q = \frac{\|\boldsymbol{s}_k^q\|_{E,2}}{\|\boldsymbol{e}_k^q\|_{E,2}}$$

  where $e_k(s_k)$: volumetric strain (stress) and $\boldsymbol{e}_k(\boldsymbol{s}_k)$: deviatoric strain (stress) associated with forward solution $\boldsymbol{u}$ and stress $\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon}[\boldsymbol{u} - \boldsymbol{w}]$.
- Correct updates B^q, G^q for possible violation of admissibility bounds.
- Morozov discrepancy criterion, if used: adjust κ to measurement noise δ through enforcement of $D(\kappa) = \delta^2$ (outer loop)

Morozov discrepancy criterion

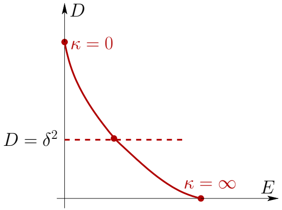
- MECR functional: $\Lambda_\kappa(\boldsymbol{X}, \boldsymbol{w}; \kappa) := \mathcal{E}(\boldsymbol{X}, \boldsymbol{w}) + \kappa \mathcal{D}(\boldsymbol{X})$ with $\boldsymbol{X} := (\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\mathcal{C}})$
- Set $E(\kappa) := \mathcal{E}(\boldsymbol{X}_\kappa, \boldsymbol{w}_\kappa),\ D(\kappa) := \mathcal{D}(\boldsymbol{X}_\kappa)$ with $(\boldsymbol{X}_\kappa, \boldsymbol{w}_\kappa) := \arg\min_{\boldsymbol{X},\boldsymbol{w}} \Lambda_\kappa(\boldsymbol{X}, \boldsymbol{w}; \kappa)$

Morozov discrepancy criterion: Seek $\boldsymbol{X}_\kappa, \kappa$ such that $D(\kappa) = \delta^2$ (δ: data noise)

Lemma: (i) $\kappa \mapsto E(\kappa)$ is increasing; (ii) $\kappa \mapsto D(\kappa)$ is decreasing.

Moreover (from limiting cases $\kappa \to 0$ and $\kappa \to \infty$):

- $D(0) > 0$ and $D(\infty) = 0$;
- $E(0) = 0$ and $E(\infty) > 0$



Consequently:

If $D(0) > \delta^2$, there exists κ such that $D(\kappa) = \delta^2$ (fulfilling Morozov's criterion)

Morozov discrepancy criterion

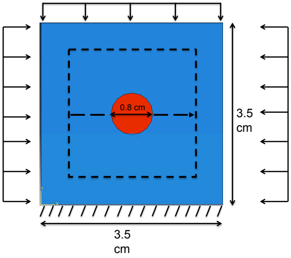
Proof of lemma: (a) we have $L'(\kappa) = \langle \partial_{\boldsymbol{X}} \mathcal{L}, \boldsymbol{X}' \rangle + \langle \partial_{\boldsymbol{w}} \mathcal{L}, \boldsymbol{w}' \rangle + \partial_\kappa \mathcal{L} = D(\kappa)$ and also $L'(\kappa) = E'(\kappa) + \kappa D'(\kappa) + D(\kappa)$; therefore $\boxed{E'(\kappa) + \kappa D'(\kappa) = 0}$.

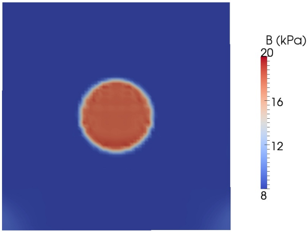
(b) We have $0 = \mathsf{d}_\kappa \big(\langle \partial_{\boldsymbol{X}} \mathcal{L}, \boldsymbol{X}' \rangle + \langle \partial_{\boldsymbol{w}} \mathcal{L}, \boldsymbol{w}' \rangle \big) = \big(L'(\kappa) - D(\kappa) \big)'$.
Moreover:

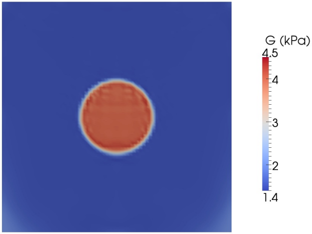
$$
\begin{aligned}
0 &= \mathsf{d}_\kappa \langle \partial_{\boldsymbol{w}} \mathcal{L}, \boldsymbol{w}' \rangle && \text{(constraint verified for any } \kappa\text{)}, \\
0 &= \mathsf{d}_\kappa \langle \partial_{\boldsymbol{X}} \mathcal{L}, \boldsymbol{X}' \rangle && \text{(stationarity eqs. verified for any } \kappa\text{)} \\
&= \langle \partial^2_{\boldsymbol{X}\boldsymbol{X}} \mathcal{L}, (\boldsymbol{X}', \boldsymbol{X}') \rangle + \langle \partial^2_{\boldsymbol{X}\boldsymbol{w}} \mathcal{L}, (\boldsymbol{X}', \boldsymbol{w}') \rangle + \langle \partial_{\boldsymbol{X}} \mathcal{L}, \boldsymbol{X}'' \rangle + D'(\kappa) \\
&= \langle \partial^2_{\boldsymbol{X}\boldsymbol{X}} \mathcal{L}, (\boldsymbol{X}', \boldsymbol{X}') \rangle + D'(\kappa)
\end{aligned}
$$

and $\langle \partial^2_{\boldsymbol{X}\boldsymbol{X}} \mathcal{L}, (\boldsymbol{X}', \boldsymbol{X}') \rangle \geq 0$. Therefore $\boxed{D'(\kappa) \leq 0}$ and, by (a), $\boxed{E'(\kappa) \geq 0}$. $\square$

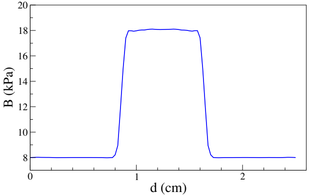
Example (2D reconstruction with 2D data and unknown BCs)

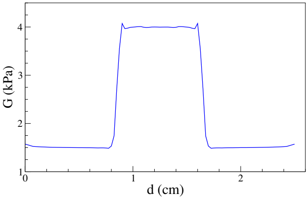






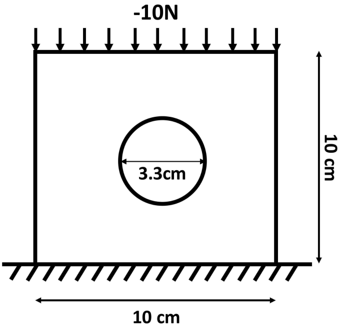
Target: $B = 18$ kPa, $G = 4$ kPa
Background: $B = 8$ kPa, $G = 1.5$ kPa
Testing frequencies 10 Hz, 30 Hz, 50 Hz, 70 Hz





Aquino, B 19

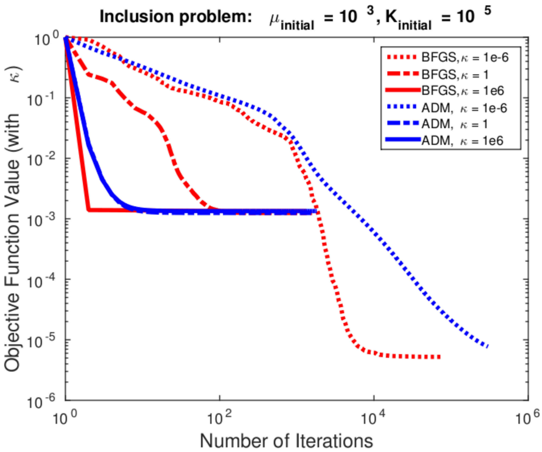
Example (2D reconstruction with 2D data): alternated dirs vs. BFGS



- Frequency = 40Hz
- $\mu_{inclusion} = 5kPa$
- $K_{inclusion} = 10Pa$
- $\mu_{background} = 1kPa$
- $K_{background} = 10Pa$
- 80x80 elements

Babaniyi, Sanders, Aquino 2017

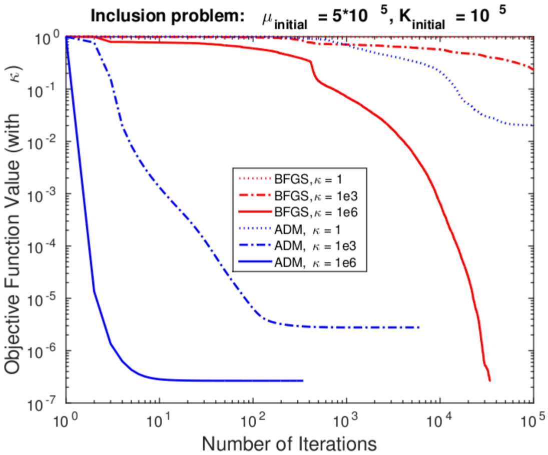
Example (2D reconstruction with 2D data): alternated dirs vs. BFGS



- 6400 optimization variables
- f = 40Hz
- $\mu_{exactIncl} = 5000$
- $\mu_{exactBack} = 1000$

Babaniyi, Sanders, Aquino 2017

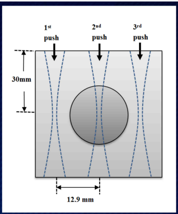
Example (2D reconstruction with 2D data): alternated dirs vs. BFGS



- 6400 optimization variables
- f = 40Hz
- $\mu_{exactIncl} = 5000$
- $\mu_{exactBack} = 1000$

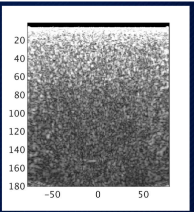
Babaniyi, Sanders, Aquino 2017

2D example (Experimental data)



Radiation Force Excitations

Experimental Setup



B-Mode Image

Assumptions:
- Bulk modulus and density equal to that of water
- Axisymmetric waves

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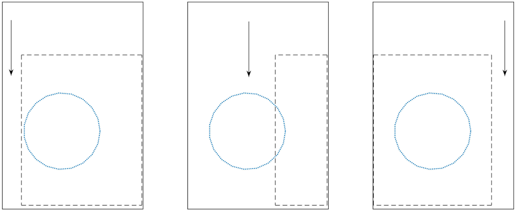
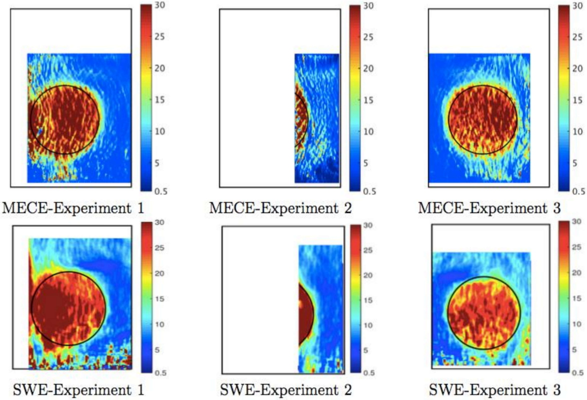


(1) (2) (3)

Aquino, Babaniyi, Bayat, Fatemi 2017

2D example (Experimental data): imaging of shear modulus



MECE-Experiment 1 MECE-Experiment 2 MECE-Experiment 3

SWE-Experiment 1 SWE-Experiment 2 SWE-Experiment 3

Experiment	Background				Inclusion			
	SWE		MECE		SWE		MECE	
No.	Mean	STD	Mean	STD	Mean	STD	Mean	STD
1	8.72	3.03	7.58	2.67	28.31	7.75	26.22	4.92
2	8.72	35.73	7.95	2.28	55.25	57.00	26.99	14.4
3	9.51	3.17	8.08	2.67	23.33	5.19	26.35	5.67
Average of 1–3	8.98		7.87		36.21		26.52	
Actual	8.33		8.33		26.67		26.67	

Aquino, Babaniyi, Bayat, Fatemi 2017

Modified ECR functional (transient, time-discrete)

Presentation after [B, Aquino, *Inverse Problems* (2014)]

- Time-stepping, $t_k = k\Delta t \ (0 \le k \le N)$
- $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{\sigma}\} := \{(\boldsymbol{u}_k, \boldsymbol{v}_k, \boldsymbol{a}_k, \boldsymbol{\sigma}_k)_{0 \le k \le N}\}$:
  time-discrete histories (displacement, velocity, acceleration, stress)
- Time-discrete MECR functional:

$$\boxed{\Lambda_\kappa(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\mathcal{C}}) := \mathcal{E}_N(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\mathcal{C}}) + \kappa \mathcal{D}_N(\boldsymbol{u})} \qquad \text{with e.g.} \quad \mathcal{D}_N(\boldsymbol{u}) := \frac{1}{2} \sum_{k=1}^{N} \big| \boldsymbol{u}_k - \boldsymbol{u}_{\text{obs},k} \big|^2$$

- Treat as constraints (i) initial and current interior balance eqns (in weak form)
  (ii) Newmark(β, γ) update relations
- Minimization of Λ_κ: requires stationarity of Lagrangian $\mathcal{L}$:

$$\begin{aligned}
\mathcal{L}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{\sigma},\ \bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{a}},\ \boldsymbol{\mathcal{C}}) := \Lambda_\kappa(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\mathcal{C}}) & \\
+ \sum_{k=0}^{N} \Big\{ \big\langle \boldsymbol{\sigma}_k, \boldsymbol{\varepsilon}[\bar{\boldsymbol{u}}_k] \big\rangle + \big\langle \rho \boldsymbol{a}_k, \bar{\boldsymbol{u}}_k \big\rangle - \mathcal{F}_k(\bar{\boldsymbol{u}}_k) \Big\} & \\
+ \sum_{k=1}^{N} \Big\{ \big\langle \big(\boldsymbol{u}_k - \boldsymbol{u}_{k-1} - \Delta t \boldsymbol{v}_{k-1} - \Delta t^2 \big[(1-\beta) \boldsymbol{a}_{k-1} + \beta \boldsymbol{a}_k \big] \big), \bar{\boldsymbol{a}}_k \big\rangle & \\
+ \big\langle \big(\boldsymbol{v}_k - \boldsymbol{v}_{k-1} - \Delta t \big[(1-\gamma) \boldsymbol{a}_{k-1} + \gamma \boldsymbol{a}_k \big] \big), \bar{\boldsymbol{v}}_k \big\rangle \Big\} &
\end{aligned}$$

- Treatment valid for (more-general) α-generalized schemes (not discussed here)
- [Allix, Feissel, Nguyen 2005]: MECR for transient 1D case

Stationarity problem

(a) $\boxed{\partial_{\boldsymbol{\sigma}_k}\mathcal{L} = \mathbf{0}}$ $\quad\Longrightarrow\quad \boldsymbol{\sigma}_k = \boldsymbol{\mathcal{C}}:\boldsymbol{\varepsilon}[\boldsymbol{u}_k - \bar{\boldsymbol{u}}_k]$

(b) $\boxed{\partial_{\bar{\boldsymbol{u}}_k}\mathcal{L} = \mathbf{0}, \quad \partial_{\bar{\boldsymbol{v}}_k}\mathcal{L} = \mathbf{0}, \quad \partial_{\bar{\boldsymbol{a}}_k}\mathcal{L} = \mathbf{0}}$ $\quad\Longrightarrow\quad$ forward problem for $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a})$

RHS depends on $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{a}})$ – **unusual!**
Newmark

(c) $\boxed{\partial_{\boldsymbol{u}_k}\mathcal{L} = \mathbf{0}, \quad \partial_{\boldsymbol{v}_k}\mathcal{L} = \mathbf{0}, \quad \partial_{\boldsymbol{a}_k} = \mathbf{0}}$ $\quad\Longrightarrow\quad$ backward problem for $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{a}})$

RHS depends on $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a})$ – usual
adjoint Newmark

(d) $\boxed{\partial_{\mathcal{C}}\mathcal{L} = \mathbf{0}}$ $\quad\Longrightarrow\quad$ Constitutive updating formulae

Newmark and adjoint Newmark obey same stability conditions.

Alternate-direction minimization

For each iteration of main minimization loop:

- partial minimization of $\Lambda_\kappa(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\mathcal{C}})$: solve (a,b,c) for $(\boldsymbol{u}^\star, \boldsymbol{v}^\star, \boldsymbol{a}^\star)$, $(\bar{\boldsymbol{u}}^\star, \bar{\boldsymbol{v}}^\star, \bar{\boldsymbol{a}}^\star)$
  coupled forward-backward problem;
- partial minimization of $\Lambda_\kappa(\boldsymbol{u}^\star, \boldsymbol{\sigma}^\star, \boldsymbol{\mathcal{C}})$: solve (d) for $\boldsymbol{\mathcal{C}}^\star$
  (analytical pointwise updating formulas $\Longrightarrow$ easy)

Coupled forward-backward problem (a,b,c) $\Longrightarrow$ major computational bottleneck

Coupled forward-backward stationarity problem

Coupled system of stationarity equations, in block form:

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B}^\mathsf{T} & -\kappa\mathbb{D} \end{bmatrix} \begin{Bmatrix} \mathbb{W} \\ \mathbb{U} \end{Bmatrix} = \begin{Bmatrix} \mathbb{F} \\ \kappa\mathbb{D}\mathbb{U}^m \end{Bmatrix},$$

where

- $\mathbb{U}^\mathsf{T} := \{(\boldsymbol{u}_0^\mathsf{T}, \boldsymbol{v}_0^\mathsf{T}, \boldsymbol{a}_0^\mathsf{T}), \ldots, (\boldsymbol{u}_N^\mathsf{T}, \boldsymbol{v}_N^\mathsf{T}, \boldsymbol{a}_N^\mathsf{T})\}$ (kinematical history);
- $\mathbb{W}^\mathsf{T} := \{(\bar{\boldsymbol{u}}_0^\mathsf{T}, \bar{\boldsymbol{v}}_0^\mathsf{T}, \bar{\boldsymbol{a}}_0^\mathsf{T}), \ldots, (\bar{\boldsymbol{u}}_N^\mathsf{T}, \bar{\boldsymbol{v}}_N^\mathsf{T}, \bar{\boldsymbol{a}}_N^\mathsf{T})\}$ (multiplier history);
- $\mathbb{F}$: applied excitation;
- $\mathbb{B}$: Newmark integrator, such that $\mathbb{B}\mathbb{U} = \mathbb{G}$ forward dynamical analysis;
  $\mathbb{B}^\mathsf{T}\mathbb{W} = \mathbb{H}$: backward dynamical analysis;
- $\mathbb{A}$: (sym. s.p.d.) coupling matrix, from ECR part of Λ_κ;
- $\mathbb{D}$: (sym. s.p.d.) "mass matrix", from L^2 measurement residual part of Λ_κ.

Kinematical constraints: $\boldsymbol{u}_k \in \mathcal{U}$, $\bar{\boldsymbol{u}}_k \in \mathcal{W}$ for all k.
Blocks $\mathbb{B}$, $\mathbb{B}^\mathsf{T}$ are square if $\mathcal{U} = \mathcal{W}$, i.e. $\Gamma_\mathsf{D} = \Gamma \setminus \Gamma_\mathsf{N}$

Stationarity problem: circumventing the coupling bottleneck

$$\begin{bmatrix} \mathbb{B} & -\mathbb{A} \\ \kappa\mathbb{D} & \mathbb{B}^{\mathsf{T}} \end{bmatrix} \begin{Bmatrix} \mathbb{U} \\ \mathbb{W} \end{Bmatrix} = \begin{Bmatrix} \mathbb{F} \\ \kappa\mathbb{D}\mathbb{U}^m \end{Bmatrix},$$

Proposed remedy to coupling bottleneck: block-SOR iterative scheme:

$$\begin{bmatrix} \mathbb{B} & 0 \\ \eta\kappa\mathbb{D} & \mathbb{B}^{\mathsf{T}} \end{bmatrix} \begin{Bmatrix} \mathbb{U}^{(i+1)} \\ \mathbb{W}^{(i+1)} \end{Bmatrix} = \begin{bmatrix} (1-\eta)\mathbb{B} & -\eta\mathbb{A} \\ 0 & (1-\eta)\mathbb{B}^{\mathsf{T}} \end{bmatrix} \begin{Bmatrix} \mathbb{U}^{(i)} \\ \mathbb{W}^{(i)} \end{Bmatrix} + \begin{Bmatrix} \eta\mathbb{F} \\ \eta\kappa\mathbb{D}\mathbb{U}^m \end{Bmatrix}.$$

($0<\eta<2$: relaxation parameter)

MB, W. Aquino, *Inverse Problems*, 2015

Convergence of block SOR algorithm

- Block-SOR and Jacobi iteration matrices ($\mathbb{U}$: stationarity solution):
$$\begin{Bmatrix} \mathbb{U}^{(i+1)} - \mathbb{U}^{\star} \\ \mathbb{W}^{(i+1)} - \mathbb{W}^{\star} \end{Bmatrix} = \mathbb{R}_{\alpha} \begin{Bmatrix} \mathbb{U}^{(i)} - \mathbb{U}^{\star} \\ \mathbb{W}^{(i)} - \mathbb{W}^{\star} \end{Bmatrix} \quad (\alpha = \mathsf{J}, \mathsf{SOR})$$
- SOR converges iff $\rho_{\mathsf{SOR}} := \rho(\mathbb{R}_{\mathsf{SOR}}) < 1$
- Eigenvalues λ of $\mathbb{R}_{\mathsf{SOR}}$ and μ of $\mathbb{R}_{\mathsf{J}}$ (simpler to evaluate) linked [Varga 62]

Proposition (MB, A. Aquino, 2015)

Let $\eta_0 := 2(1+\rho_{\mathsf{J}})^{-1}$. Then, $\rho_{\mathsf{SOR}}(\eta) < 1$ for any $\eta \in]0, \eta_0[$. Moreover:

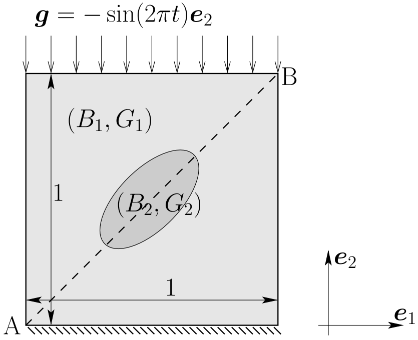
(a) $\min_{\eta \in]0,\eta_0[} \rho_{\mathsf{SOR}}(\eta) = 1 - \eta_1$ with $\eta_1 = 2 \big/ \big(1 + (1+\rho_{\mathsf{J}}^2)^{1/2}\big)$

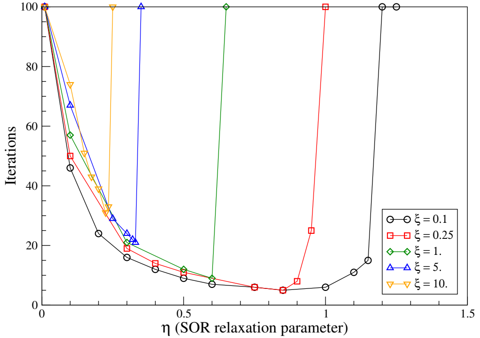
(b) $\rho_{\mathsf{J}} = O(\kappa^{1/2})$, and hence $\lim_{\kappa \to \infty} \eta_0 = 0, \quad \lim_{\kappa \to \infty} \min_{\eta \in]0,\eta_0[} \rho_{\mathsf{SOR}}(\eta) = 1$

κ large (suitable for accurate data) (i) narrows convergence interval $]0, \eta_0[$
(ii) increases block-SOR iteration count

2D example (block SOR assessment)

- Load: time-harmonic pressure on top side (duration 1s, freq. 1Hz);
  bottom side clamped
- $(B_1, G_1) = (3, 2)$; $(B_2, G_2) = (6, 4)$
- Full-field measured displacement for 1s duration
- Time step: $\Delta t = 0.01s$
- Meshes: 13,122 nodes (reconstruction, regular mesh),
  19,216 nodes (data generation)

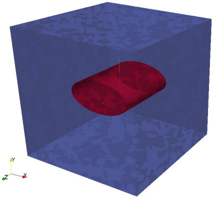


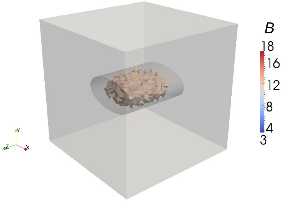


Simulations by WA, using the Sierra/SDA code of SANDIA Natl. Labs.

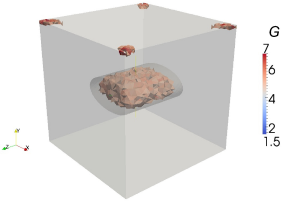
3D example (synthetic data)

- Load: time-harmonic pressure on top and side faces (duration 1s, freq. 1Hz); bottom face clamped
- Full-field measured displacement for 1s duration
- Time step: $\Delta t = 0.01s$
- Meshes: 50,000 nodes (reconstruction, regular mesh), 75,000 nodes (data generation)
- 550,000 unknown moduli



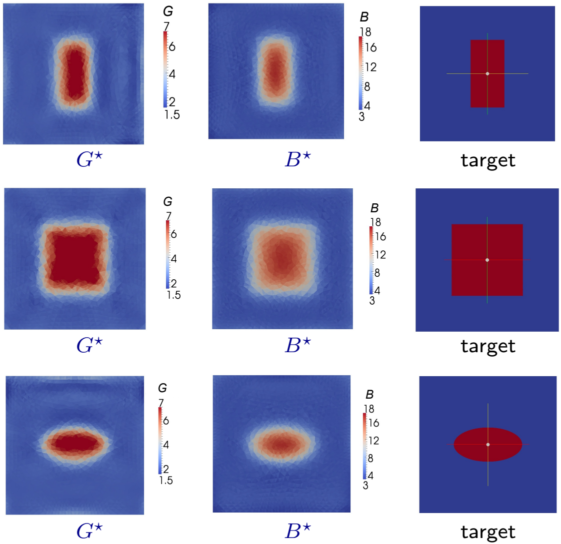


$B^{\star}$



$G^{\star}$

3D example (Synthetic data)

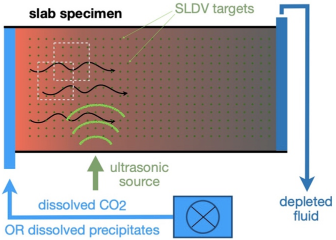


- About 200 MECR iterations;
- At most 5 SOR iterations per MECE iteration

Introduction

This part follows [B, Salasiya, Guzina; JMPS (2024)]

- Material characterization of lossy solids treated as linear viscoelastic: applications e.g.
  - ▷ Account for lossy biological media in elastography
  - ▷ Geomechanics, geophysics (dissipation linked to e.g. hydrocarbon reservoir parameters)



Viscoelastic characterization of rock specimens undergoing carbonation, excited under ultrasonically plane stress condition.

- Use of interior data feasible: ultrasound, MRI, laser vibrometer in rock mechanics
- This work:
  - ▷ Extension to linear viscoelasticity of elastodynamic MECR with missing BC info
    Focus on case with interior measurement and no BC information
  - ▷ Both transient and time-harmonic cases treated

This work, US side funded by Center on Geo-processes in Mineral Carbon Storage, US Dept. of Energy

Linear viscoelastic solid as a generalized standard material

- Standard generalized material format for linear viscoelasticity:

$$\boldsymbol{\sigma}[\boldsymbol{u}] = \boldsymbol{\sigma}^{\mathrm{e}}[\boldsymbol{u}] + \boldsymbol{\sigma}^{\mathrm{v}}[\boldsymbol{u}]$$

$$\boldsymbol{\sigma}^{\mathrm{e}}[\boldsymbol{u}] = \partial_{\varepsilon}\psi, \qquad \boldsymbol{\sigma}^{\mathrm{v}}[\boldsymbol{u}] = \partial_{\dot{\varepsilon}}\varphi, \qquad \boldsymbol{A}[\boldsymbol{u}] = -\partial_{\alpha}\psi = \partial_{\dot{\alpha}}\varphi.$$

- Free-energy potential and dissipation potential of general form (must be convex)

$$\boxed{\begin{aligned}\psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) &= \tfrac{1}{2}\big(\boldsymbol{\varepsilon}:\boldsymbol{\mathcal{C}}_{\varepsilon}:\boldsymbol{\varepsilon} + 2\boldsymbol{\varepsilon}:\boldsymbol{\mathcal{C}}_{\mathrm{m}}:\boldsymbol{\alpha} + \boldsymbol{\alpha}:\boldsymbol{\mathcal{C}}_{\alpha}:\boldsymbol{\alpha}\big),\\ \varphi(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}}) &= \tfrac{1}{2}\big(\dot{\boldsymbol{\varepsilon}}:\boldsymbol{\mathcal{D}}_{\varepsilon}:\dot{\boldsymbol{\varepsilon}} + 2\dot{\boldsymbol{\varepsilon}}:\boldsymbol{\mathcal{D}}_{\mathrm{m}}:\dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{\alpha}}:\boldsymbol{\mathcal{D}}_{\alpha}:\dot{\boldsymbol{\alpha}}\big),\end{aligned}}$$

$$\mathbf{p} = \boldsymbol{\mathcal{C}}_{\varepsilon}, \boldsymbol{\mathcal{C}}_{\alpha}, \boldsymbol{\mathcal{D}}_{\varepsilon} \ldots$$

  - ▷ $\boldsymbol{\alpha}$: “viscoelastic strain” internal variable,
  - ▷ $\boldsymbol{\mathcal{C}}_{\varepsilon}, \boldsymbol{\mathcal{C}}_{\alpha}, \boldsymbol{\mathcal{D}}_{\varepsilon}, \boldsymbol{\mathcal{D}}_{\alpha}$: 4th order tensors (maj + min symm., define positive quadratic forms,
  - ▷ $\boldsymbol{\mathcal{C}}_{\mathrm{m}}, \boldsymbol{\mathcal{D}}_{\mathrm{m}}$: 4th order tensors (min. symm., maj. symm. for convenience), s.t. $\psi, \varphi \geq 0$.

- Viscoelastic strain

$$\partial_{\alpha}\psi + \partial_{\dot{\alpha}}\varphi = \mathbf{0} \quad \Longrightarrow \quad \boxed{\boldsymbol{\alpha}[\boldsymbol{u}](t) = -\boldsymbol{\mathcal{D}}_{\alpha}^{-1}:\boldsymbol{\mathcal{D}}_{\mathrm{m}}:\boldsymbol{\varepsilon}(t) - \mathsf{F}[\widehat{\boldsymbol{\mathcal{C}}}:\boldsymbol{\varepsilon}](t)}$$

$$\widehat{\boldsymbol{\mathcal{C}}} := \boldsymbol{\mathcal{C}}_{\mathrm{m}} - \boldsymbol{\mathcal{C}}_{\alpha}:\boldsymbol{\mathcal{D}}_{\alpha}^{-1}:\boldsymbol{\mathcal{D}}_{\mathrm{m}}, \qquad \mathsf{F}[\boldsymbol{s}](t) = \int_0^t \exp[-\boldsymbol{\mathcal{D}}_{\alpha}^{-1}:\boldsymbol{\mathcal{C}}_{\alpha}(t-\tau)]:\boldsymbol{\mathcal{D}}_{\alpha}^{-1}:\boldsymbol{s}(\tau)\,\mathsf{d}\tau$$

- Stress-strain relation:

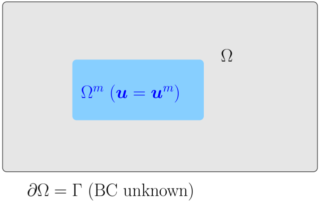
$$\boldsymbol{\sigma}[\boldsymbol{u}](t) = \boldsymbol{\mathcal{C}}_{\mathsf{I}}:\boldsymbol{\varepsilon}(t) + \boldsymbol{\mathcal{D}}_{\mathsf{I}}:\dot{\boldsymbol{\varepsilon}}(t) - \widehat{\boldsymbol{\mathcal{C}}}^{\mathsf{T}}:\mathsf{F}[\widehat{\boldsymbol{\mathcal{C}}}:\boldsymbol{\varepsilon}](t)$$

  with instantaneous tensors

$$\boldsymbol{\mathcal{C}}_{\mathsf{I}} = \boldsymbol{\mathcal{C}}_{\varepsilon} - \boldsymbol{\mathcal{C}}_{\mathrm{m}}:\boldsymbol{\mathcal{C}}_{\alpha}^{-1}:\boldsymbol{\mathcal{C}}_{\mathrm{m}} + \widehat{\boldsymbol{\mathcal{C}}}^{\mathsf{T}}:\boldsymbol{\mathcal{C}}_{\alpha}^{-1}:\widehat{\boldsymbol{\mathcal{C}}}, \qquad \boldsymbol{\mathcal{D}}_{\mathsf{I}} = \boldsymbol{\mathcal{D}}_{\varepsilon} - \boldsymbol{\mathcal{D}}_{\mathrm{m}}:\boldsymbol{\mathcal{D}}_{\alpha}^{-1}:\boldsymbol{\mathcal{D}}_{\mathrm{m}}$$

Inverse problem

- Identify (homogeneous or heterogeneous) VE parameters $\mathbf{p} = \boldsymbol{\mathcal{C}}_\varepsilon, \boldsymbol{\mathcal{C}}_\alpha, \boldsymbol{\mathcal{D}}_\varepsilon \ldots$ from interior kinematic data $\boldsymbol{u}_{\text{obs}}(\cdot, t)$ or $\boldsymbol{u}_{\text{obs}}(\cdot, \omega)$



- This work: MECR-based PDE-constrained approach

$$\boxed{\min_{\boldsymbol{u},\boldsymbol{\sigma},\mathbf{p}} \mathcal{E}(\boldsymbol{u}, \boldsymbol{\sigma}, \mathbf{p}) + \kappa \mathcal{D}_T(\boldsymbol{u} - \boldsymbol{u}_{\text{obs}}) \qquad \text{s.t.} \quad \boldsymbol{u} \text{ KA}, \boldsymbol{\sigma} \text{ DA}} \qquad \mathcal{D}_T(\boldsymbol{v}) = \frac{1}{2}\int_0^T \mathcal{D}(\boldsymbol{v}(t))\, \mathrm{d}t$$

- Conjugate potentials:

$$\psi^\star(\boldsymbol{\sigma}^{\mathrm{e}}, \boldsymbol{A}) = \max_{\varepsilon,\alpha} \big[\boldsymbol{\sigma}^{\mathrm{e}} : \boldsymbol{\varepsilon} - \boldsymbol{A} : \boldsymbol{\alpha} - \psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) \big],$$
$$\varphi^\star(\boldsymbol{\sigma}^{\mathrm{v}}, \boldsymbol{A}) = \max_{\dot{\varepsilon},\dot{\alpha}} \big[\boldsymbol{\sigma}^{\mathrm{v}} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{A} : \dot{\boldsymbol{\alpha}} - \varphi(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\alpha}}) \big].$$

- Legendre-Fenchel gaps:

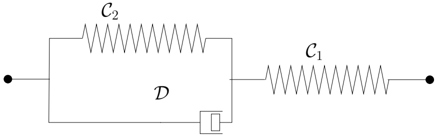
$$\epsilon_\psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \boldsymbol{\sigma}^{\mathrm{e}}, \boldsymbol{A}) := \psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) + \psi^\star(\boldsymbol{\sigma}^{\mathrm{e}}, \boldsymbol{A}) - \boldsymbol{\sigma}^{\mathrm{e}} : \boldsymbol{\varepsilon} + \boldsymbol{A} : \boldsymbol{\alpha} \qquad \geqslant 0,$$
$$\epsilon_\varphi(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\alpha}}, \boldsymbol{\sigma}^{\mathrm{v}}, \boldsymbol{A}) := \varphi(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\alpha}}) + \varphi^\star(\boldsymbol{\sigma}^{\mathrm{v}}, \boldsymbol{A}) - \boldsymbol{\sigma}^{\mathrm{v}} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{A} : \dot{\boldsymbol{\alpha}} \qquad \geqslant 0.$$

- Chosen pointwise ECR: $e_{\text{ECR}}(\boldsymbol{x}, t) = \epsilon_\psi(\boldsymbol{x}, t) + T \epsilon_\varphi(\boldsymbol{x}, t)$.

Example: standard linear solid

$$\boldsymbol{\mathcal{C}}_\varepsilon = -\boldsymbol{\mathcal{C}}_{\rm m} = \boldsymbol{\mathcal{C}}_1, \qquad \boldsymbol{\mathcal{C}}_\alpha = \boldsymbol{\mathcal{C}}_1 + \boldsymbol{\mathcal{C}}_2, \qquad \boldsymbol{\mathcal{D}}_\alpha = \boldsymbol{\mathcal{D}}, \qquad \boldsymbol{\mathcal{D}}_\varepsilon = \boldsymbol{\mathcal{D}}_{\rm m} = \mathbf{0},$$

$$\boxed{\begin{aligned} &\psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) = \tfrac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha}):\boldsymbol{\mathcal{C}}_1:(\boldsymbol{\varepsilon}-\boldsymbol{\alpha}) + \frac{1}{2}\boldsymbol{\alpha}:\boldsymbol{\mathcal{C}}_2:\boldsymbol{\alpha}, &&\varphi(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}}) = \tfrac{1}{2}\dot{\boldsymbol{\alpha}}:\boldsymbol{\mathcal{D}}:\dot{\boldsymbol{\alpha}},\\ &\psi^\star(\boldsymbol{\sigma},\boldsymbol{A}) = \tfrac{1}{2}(\boldsymbol{\sigma}-\boldsymbol{A}):\boldsymbol{\mathcal{C}}_2^{-1}:(\boldsymbol{\sigma}-\boldsymbol{A}) + \tfrac{1}{2}\boldsymbol{\sigma}:\boldsymbol{\mathcal{C}}_1^{-1}:\boldsymbol{\sigma}, &&\varphi^\star(\boldsymbol{A}) = \tfrac{1}{2}\boldsymbol{A}:\boldsymbol{\mathcal{D}}^{-1}:\boldsymbol{A}.\end{aligned}}$$



$$\boldsymbol{\sigma}[\boldsymbol{u}] = \boldsymbol{\sigma}^{\rm e}[\boldsymbol{u}] = \boldsymbol{\mathcal{C}}_1:(\boldsymbol{\varepsilon}-\boldsymbol{\alpha}), \quad \boldsymbol{A}[\boldsymbol{u}] = \boldsymbol{\mathcal{C}}_1:(\boldsymbol{\varepsilon}-\boldsymbol{\alpha}) - \boldsymbol{\mathcal{C}}_2:\boldsymbol{\alpha} = \boldsymbol{\mathcal{D}}_\alpha:\dot{\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha}(t) = \mathsf{F}[\boldsymbol{\mathcal{C}}_1:\boldsymbol{\varepsilon}](t)$$

Pointwise Legendre-Fenchel gaps:

$$\boxed{\begin{aligned} e_\psi &= \tfrac{1}{2}\big(\boldsymbol{\sigma}-\boldsymbol{\mathcal{C}}_1:(\boldsymbol{\varepsilon}-\boldsymbol{\alpha})\big):\boldsymbol{\mathcal{C}}_1^{-1}:\big(\boldsymbol{\sigma}-\boldsymbol{\mathcal{C}}_1:(\boldsymbol{\varepsilon}-\boldsymbol{\alpha})\big) + \tfrac{1}{2}(\boldsymbol{\sigma}-\boldsymbol{A}-\boldsymbol{\mathcal{C}}_2:\boldsymbol{\alpha}):\boldsymbol{\mathcal{C}}_2^{-1}:(\boldsymbol{\sigma}-\boldsymbol{A}-\boldsymbol{\mathcal{C}}_2:\boldsymbol{\alpha}))\\ e_\varphi &= \tfrac{1}{2}\big(\boldsymbol{A}-\boldsymbol{\mathcal{D}}:\dot{\boldsymbol{\alpha}}\big):\boldsymbol{\mathcal{D}}^{-1}:\big(\boldsymbol{A}-\boldsymbol{\mathcal{D}}:\dot{\boldsymbol{\alpha}}\big).\end{aligned}}$$

MECR-based minimization

- Displacement spaces:

$$\mathcal{U} := \mathcal{V}, \qquad \mathcal{W} := \big\{ \boldsymbol{v} \in \mathcal{V},\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma \big\} \subset \mathcal{U}$$

$\mathcal{V}$: energy space, e.g. $\mathcal{V} = H^1(\Omega \times [0,T]; \mathbb{R}^d)$ (transient), $\mathcal{V} = H^1(\Omega; \mathbb{R}^d)$ (time-harmonic)

- Interior balance of linear momentum, weak form (interior equations only):

$$\int_\Omega \int_0^T \big(\boldsymbol{\sigma} : \boldsymbol{\varepsilon}[\boldsymbol{w}] + \rho \ddot{\boldsymbol{u}} \cdot \boldsymbol{w} \big) \, \text{d}t \, \text{d}V = 0 \qquad \forall\, \boldsymbol{w} \in \mathcal{W}.$$

- Overall constitutive mismatch for given $\mathbf{p}$ (with $\mathbf{X} := (\boldsymbol{u}, \boldsymbol{\alpha}, \boldsymbol{\sigma}^{\text{e}}, \boldsymbol{\sigma}^{\text{v}}, \boldsymbol{A})$):

$$\boxed{\mathcal{E}(\mathbf{X}, \mathbf{p}) \ = \ \mathcal{E}^{\text{e}}(\boldsymbol{\varepsilon}[\boldsymbol{u}], \boldsymbol{\alpha}, \boldsymbol{\sigma}^{\text{e}}, \boldsymbol{A}, \mathbf{p}) \ + \ \mathcal{E}^{\text{v}}(\dot{\boldsymbol{\varepsilon}}[\boldsymbol{u}], \dot{\boldsymbol{\alpha}}, \boldsymbol{\sigma}^{\text{v}}, \boldsymbol{A}, \mathbf{p})},$$

$$\mathcal{E}^{\text{e}} := \int_\Omega \int_0^T \epsilon_\psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \boldsymbol{\sigma}^{\text{e}}, \boldsymbol{A}, \mathbf{p}) \, \text{d}t \, \text{d}V, \qquad \mathcal{E}^{\text{v}} := \int_\Omega \int_0^T T \epsilon_\varphi(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\alpha}}, \boldsymbol{\sigma}^{\text{v}}, \boldsymbol{A}, \mathbf{p}) \, \text{d}t \, \text{d}V.$$

- Modified ECR (MECR) functional:

$$\boxed{\Lambda_\kappa(\mathbf{X}, \mathbf{p}) := \mathcal{E}(\mathbf{X}, \mathbf{p}) + \tfrac{1}{2} \kappa \mathcal{D}_T(\boldsymbol{u} - \boldsymbol{u}_{\text{obs}}, \boldsymbol{u} - \boldsymbol{u}_{\text{obs}}),}$$

- Find compromise $\mathbf{p}, \mathbf{X}$ (constitutive equations vs. data reproduction): PDE-constrained minimization

$$\min_{\mathbf{X}, \mathbf{p}} \Lambda_\kappa(\mathbf{X}, \mathbf{p}) \qquad \text{subject to } \boldsymbol{u} \in \mathcal{U} \ \text{ and } \ \text{(weak) balance eq.}\ .$$

1st-order optimality conditions of MECR functional

$\min_{\mathsf{X},\mathsf{p}} \Lambda_\kappa(\mathsf{X},\mathsf{p})$ subject to $\boldsymbol{u}\in\mathcal{U}$ and (weak) balance eq. .

- Lagrangian (Lagrange multiplier $\boldsymbol{w} \in \mathcal{W}$):

$$\mathcal{L}(\mathsf{X},\boldsymbol{w},\mathsf{p}) := \Lambda_\kappa(\mathsf{X},\mathsf{p}) - \int_\Omega \int_0^T \big((\boldsymbol{\sigma}^{\rm e} + \boldsymbol{\sigma}^{\rm v}) : \boldsymbol{\varepsilon}[\boldsymbol{w}] + \rho \ddot{\boldsymbol{u}} \cdot \boldsymbol{w} \big) \, \mathrm{d}t \, \mathrm{d}V.$$

- 1st-order optimality conditions: stress and internal variables:

(a) $\big\langle \partial_{\sigma^{\rm e}} \mathcal{L}, \widehat{\boldsymbol{\sigma}}^{\rm e} \big\rangle = 0 \quad \forall\, \widehat{\boldsymbol{\sigma}}^{\rm e}$, (c) $\big\langle \partial_{A} \mathcal{L}, \widehat{\boldsymbol{A}} \big\rangle = 0 \quad \forall\, \widehat{\boldsymbol{A}}$,

(b) $\big\langle \partial_{\sigma^{\rm v}} \mathcal{L}, \widehat{\boldsymbol{\sigma}}^{\rm v} \big\rangle = 0 \quad \forall\, \widehat{\boldsymbol{\sigma}}^{\rm v}$, (d) $\big\langle \partial_{\alpha} \mathcal{L}, \widehat{\boldsymbol{\alpha}} \big\rangle = 0 \quad \forall\, \widehat{\boldsymbol{\alpha}}$.

1st-order optimality conditions: kinematical variables:

(a) $\big\langle \partial_{w} \mathcal{L}, \widehat{\boldsymbol{w}} \big\rangle = 0 \quad \forall\, \widehat{\boldsymbol{w}}$,

(b) $\big\langle \partial_{u} \mathcal{L}, \widehat{\boldsymbol{u}} \big\rangle = 0 \quad \forall\, \widehat{\boldsymbol{u}}$,

1st-order optimality conditions: material parameters:

$\big\langle \partial_{\mathsf{p}} \mathcal{L}, \widehat{\mathsf{p}} \big\rangle = 0 \quad \forall\, \widehat{\mathsf{p}}$.

1st-order optimality conditions: local equations

Local stationarity equations ($\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}[\boldsymbol{u}]$, $\boldsymbol{\eta} \equiv \boldsymbol{\varepsilon}[\boldsymbol{w}]$)

(a) $0 = \int_\Omega \int_0^T \Big\{ \partial_{\sigma^{\rm e}} \psi^\star - \boldsymbol{\varepsilon} - \boldsymbol{\eta} \Big\} : \widehat{\boldsymbol{\sigma}}^{\rm e} \, \text{d}t \, \text{d}V \qquad \forall \widehat{\boldsymbol{\sigma}}^{\rm e},$

(b) $0 = \int_\Omega \int_0^T \Big\{ T\big(\partial_{\sigma^{\rm v}} \varphi^\star - \dot{\boldsymbol{\varepsilon}}\big) - \boldsymbol{\eta} \Big\} : \widehat{\boldsymbol{\sigma}}^{\rm v} \, \text{d}t \, \text{d}V \qquad \forall \widehat{\boldsymbol{\sigma}}^{\rm v},$

(c) $0 = \int_\Omega \int_0^T \Big\{ \partial_{A} \psi^\star + \boldsymbol{\alpha} + T\big(\partial_{A} \varphi^\star - \dot{\boldsymbol{\alpha}}\big) \Big\} : \widehat{\boldsymbol{A}} \, \text{d}t \, \text{d}V \qquad \forall \widehat{\boldsymbol{A}},$

(d) $0 = \int_\Omega \int_0^T \Big\{ \big(\partial_{\alpha} \psi + \boldsymbol{A}\big) : \widehat{\boldsymbol{\alpha}} + T\big(\partial_{\dot{\alpha}} \varphi - \boldsymbol{A}\big) : \dot{\widehat{\boldsymbol{\alpha}}} \Big\} \, \text{d}t \, \text{d}V \qquad \forall \widehat{\boldsymbol{\alpha}},\ \widehat{\boldsymbol{\alpha}}(\cdot,0) = \boldsymbol{0}.$

Closed-form solution:

$$
\text{(a), (b), (c)} \quad \Longrightarrow \quad \boxed{\begin{cases} \boldsymbol{\sigma}^{\rm e} = \boldsymbol{\mathcal{C}}_\varepsilon : \boldsymbol{\varepsilon} + \boldsymbol{\mathcal{C}}_{\rm m} : \boldsymbol{\alpha} + (\boldsymbol{\mathcal{C}}_\varepsilon^{\sf S} + \boldsymbol{\mathcal{C}}_{\rm m} : \boldsymbol{\mathcal{C}}_\alpha^{-1} : \widehat{\boldsymbol{\mathcal{C}}}) : \boldsymbol{\eta} - \boldsymbol{\mathcal{C}}_{\rm m} : \boldsymbol{\beta}, \\ \boldsymbol{\sigma}^{\rm v} = \boldsymbol{\mathcal{D}}_{\rm m} : \dot{\boldsymbol{\alpha}} + \boldsymbol{\mathcal{D}}_\varepsilon : \dot{\boldsymbol{\varepsilon}} + \frac{1}{T} \boldsymbol{\mathcal{D}}_{\sf I} : \boldsymbol{\eta} - \frac{1}{T} \boldsymbol{\mathcal{D}}_{\rm m} : \boldsymbol{\beta}, \\ \boldsymbol{A} = \boldsymbol{\mathcal{D}}_\alpha : \dot{\boldsymbol{\alpha}} + \boldsymbol{\mathcal{D}}_{\rm m} : \dot{\boldsymbol{\varepsilon}} - \frac{1}{T} \boldsymbol{\mathcal{D}}_\alpha : \boldsymbol{\beta}, \end{cases}}
$$

$$
\big(\tfrac{1}{T} \boldsymbol{\mathcal{D}}_\alpha + \boldsymbol{\mathcal{C}}_\alpha\big) : \boldsymbol{\beta} = \boldsymbol{\mathcal{D}}_\alpha : \dot{\boldsymbol{\alpha}} + \boldsymbol{\mathcal{D}}_{\rm m} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\mathcal{C}}_\alpha : \boldsymbol{\alpha} + \boldsymbol{\mathcal{C}}_{\rm m} : \boldsymbol{\varepsilon} + \widehat{\boldsymbol{\mathcal{C}}} : \boldsymbol{\eta}.
$$

(d) $\Longrightarrow \quad 0 = \partial_\alpha \psi + \boldsymbol{A} - T \partial_t \big(\partial_{\dot{\alpha}} \varphi - \boldsymbol{A}\big), \qquad 0 = \big(\partial_{\dot{\alpha}} \varphi - \boldsymbol{A}\big)(T),$

$\Longrightarrow \quad \boldsymbol{\mathcal{D}}_\alpha : \dot{\boldsymbol{\beta}} - \boldsymbol{\mathcal{C}}_\alpha : \boldsymbol{\beta} + \widehat{\boldsymbol{\mathcal{C}}} : \boldsymbol{\eta} = \boldsymbol{0}, \qquad \boldsymbol{\beta}(T) = 0$ using (a)-(c)

$\Longrightarrow \quad \boxed{\boldsymbol{\beta}(t) = \mathsf{F}_{\sf R}[\widehat{\boldsymbol{\mathcal{C}}} : \boldsymbol{\eta}_{\sf R}](t)} \qquad \big(f_{\sf R}(t) = f(T-t)\big)$

1st-order optimality conditions: local equations

(a), (b), (c) $\Longrightarrow$
$$\boxed{\begin{cases} \boldsymbol{\sigma}^{\mathrm{e}} = \boldsymbol{\mathcal{C}}_{\varepsilon} : \boldsymbol{\varepsilon} + \boldsymbol{\mathcal{C}}_{\mathrm{m}} : \boldsymbol{\alpha} + (\boldsymbol{\mathcal{C}}_{\varepsilon}^{\mathsf{S}} + \boldsymbol{\mathcal{C}}_{\mathrm{m}} : \boldsymbol{\mathcal{C}}_{\alpha}^{-1} : \widehat{\boldsymbol{\mathcal{C}}}) : \boldsymbol{\eta} - \boldsymbol{\mathcal{C}}_{\mathrm{m}} : \boldsymbol{\beta}, \\ \boldsymbol{\sigma}^{\mathrm{v}} = \boldsymbol{\mathcal{D}}_{\mathrm{m}} : \dot{\boldsymbol{\alpha}} + \boldsymbol{\mathcal{D}}_{\varepsilon} : \dot{\boldsymbol{\varepsilon}} + \frac{1}{T} \boldsymbol{\mathcal{D}}_{\mathsf{I}} : \boldsymbol{\eta} - \frac{1}{T} \boldsymbol{\mathcal{D}}_{\mathrm{m}} : \boldsymbol{\beta}, \\ \boldsymbol{A} = \boldsymbol{\mathcal{D}}_{\alpha} : \dot{\boldsymbol{\alpha}} + \boldsymbol{\mathcal{D}}_{\mathrm{m}} : \dot{\boldsymbol{\varepsilon}} - \frac{1}{T} \boldsymbol{\mathcal{D}}_{\alpha} : \boldsymbol{\beta}, \end{cases}}$$

$$\big(\tfrac{1}{T}\boldsymbol{\mathcal{D}}_{\alpha} + \boldsymbol{\mathcal{C}}_{\alpha}\big) : \boldsymbol{\beta} = \boldsymbol{\mathcal{D}}_{\alpha} : \dot{\boldsymbol{\alpha}} + \boldsymbol{\mathcal{D}}_{\mathrm{m}} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\mathcal{C}}_{\alpha} : \boldsymbol{\alpha} + \boldsymbol{\mathcal{C}}_{\mathrm{m}} : \boldsymbol{\varepsilon} + \widehat{\boldsymbol{\mathcal{C}}} : \boldsymbol{\eta}. \qquad (\star)$$

(d) $\Longrightarrow$ $\boxed{\boldsymbol{\beta}(t) = \mathsf{F}_{\mathsf{R}}[\widehat{\boldsymbol{\mathcal{C}}} : \boldsymbol{\eta}_{\mathsf{R}}](t)}$

- Use $\boldsymbol{\beta}$ in $(\star)$, to obtain
  $$\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}[\boldsymbol{u}](t) + \mathsf{F}[\boldsymbol{\mathcal{D}}_{\alpha} : (\tfrac{1}{T}\boldsymbol{\beta} + \dot{\boldsymbol{\beta}})](t)$$
  Finally, evaluate $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{e}} + \boldsymbol{\sigma}^{\mathrm{v}}$ to find
  $$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}[\boldsymbol{u}](t) + \boldsymbol{\mathcal{S}}_t[\boldsymbol{w}](t) \qquad \boldsymbol{\mathcal{S}}_t[\boldsymbol{w}] = (\boldsymbol{\mathcal{C}}_{\mathsf{I}} + \tfrac{1}{T}\boldsymbol{\mathcal{D}}_{\mathsf{I}}) : \boldsymbol{\eta} + \widehat{\boldsymbol{\mathcal{C}}}^{\mathsf{T}} : \big(\boldsymbol{\alpha} - \boldsymbol{\alpha}[\boldsymbol{u}] - \boldsymbol{\beta}\big).$$
- Key property:
  $$\boxed{\int_0^T \boldsymbol{\mathcal{S}}_t[\boldsymbol{w}] : \boldsymbol{\eta} \, \mathrm{d}t = \int_0^T \Big\{ \boldsymbol{\eta} : \big(\boldsymbol{\mathcal{C}}_{\varepsilon}^{\mathsf{S}} + \tfrac{1}{T}\boldsymbol{\mathcal{D}}_{\mathsf{I}}\big) : \boldsymbol{\eta} + \tfrac{1}{T}\boldsymbol{\beta} : \boldsymbol{\mathcal{D}}_{\alpha} : \boldsymbol{\beta} + \dot{\boldsymbol{\beta}} : \boldsymbol{\mathcal{D}}_{\alpha} : \boldsymbol{\mathcal{C}}_{\alpha}^{-1} : \boldsymbol{\mathcal{D}}_{\alpha} : \dot{\boldsymbol{\beta}} \Big\} \, \mathrm{d}t \geqslant 0.}$$

Purely elastic case much simpler: $\boxed{\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}[\boldsymbol{u}](t) + \boldsymbol{\sigma}[\boldsymbol{w}](t) = \boldsymbol{\mathcal{C}}_{\varepsilon} : \boldsymbol{\varepsilon}[\boldsymbol{u} + \boldsymbol{w}]}$ sole local eqn.

1st-order optimality conditions: global equations

- A priori:

$$\int_\Omega \int_0^T \big(\boldsymbol{\sigma} : \boldsymbol{\varepsilon}[\widehat{\boldsymbol{w}}] + \rho \ddot{\boldsymbol{u}} \cdot \widehat{\boldsymbol{w}} \big) \, \mathrm{d}t \, \mathrm{d}V = 0 \qquad \forall\, \widehat{\boldsymbol{w}} \in \mathcal{W}.$$

$$\int_\Omega \int_0^T \Big\{ \big(\partial_\varepsilon \psi - \boldsymbol{\sigma}^{\mathrm{e}} \big) : \boldsymbol{\varepsilon}[\widehat{\boldsymbol{u}}] + T \big(\partial_{\dot{\varepsilon}} \varphi - \boldsymbol{\sigma}^{\mathrm{v}} \big) : \boldsymbol{\varepsilon}[\dot{\widehat{\boldsymbol{u}}}] - \rho \boldsymbol{w} \cdot \ddot{\widehat{\boldsymbol{u}}} \Big\} \, \mathrm{d}t \, \mathrm{d}V$$
$$+ \kappa \int_0^T \mathcal{D}(\boldsymbol{u} - \boldsymbol{u}_{\mathsf{obs}}, \widehat{\boldsymbol{u}}) \, \mathrm{d}t = 0 \qquad \forall\, \widehat{\boldsymbol{u}} \in \mathcal{U}.$$

- Use $\boldsymbol{\sigma} = \boldsymbol{\sigma}[\boldsymbol{u}] + \boldsymbol{\mathcal{S}}_t[\boldsymbol{w}]$, integrate by parts in time, use results from local stationarity cnds and reciprocity identity $\int_0^T \big(\boldsymbol{\sigma}[\boldsymbol{u}] : \boldsymbol{\varepsilon}[\boldsymbol{w}] - \boldsymbol{\sigma}_{\mathsf{R}}[\boldsymbol{w}_{\mathsf{R}}] : \boldsymbol{\varepsilon}[\boldsymbol{u}] \big) \, \mathrm{d}t \;=\; \big(\boldsymbol{\varepsilon}[\boldsymbol{w}] : \boldsymbol{\mathcal{D}}_{\mathsf{I}} : \boldsymbol{\varepsilon}[\boldsymbol{u}] \big) \big|_0^T$.

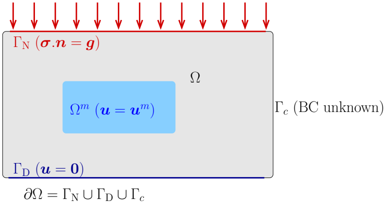
  Obtain forward-backward (underdetermined / overdetermined) stationarity system for $\boldsymbol{u}, \boldsymbol{w}$:

$$\int_\Omega \int_0^T \big(\boldsymbol{\sigma}[\boldsymbol{u}] : \boldsymbol{\varepsilon}[\widehat{\boldsymbol{w}}] + \rho \ddot{\boldsymbol{u}} \cdot \widehat{\boldsymbol{w}} \big) \, \mathrm{d}t \, \mathrm{d}V + \int_\Omega \int_0^T \boldsymbol{\mathcal{S}}_t[\boldsymbol{w}] : \boldsymbol{\varepsilon}[\widehat{\boldsymbol{w}}] \, \mathrm{d}t \, \mathrm{d}V = 0 \qquad \forall\, \widehat{\boldsymbol{w}} \in \mathcal{W},$$
$$\boldsymbol{u}(\cdot, 0) = \dot{\boldsymbol{u}}(\cdot, 0) = \boldsymbol{0} \qquad \text{in } \Omega,$$
$$\int_\Omega \int_0^T \big(\boldsymbol{\sigma}_{\mathsf{R}}[\boldsymbol{w}_{\mathsf{R}}] : \boldsymbol{\varepsilon}[\widehat{\boldsymbol{u}}] + \rho \ddot{\boldsymbol{w}} \cdot \widehat{\boldsymbol{u}} \big) \, \mathrm{d}t \, \mathrm{d}V - \kappa \int_0^T \mathcal{D}(\boldsymbol{u}, \widehat{\boldsymbol{u}}) \, \mathrm{d}t = -\kappa \int_0^T \mathcal{D}(\boldsymbol{u}_{\mathsf{obs}}, \widehat{\boldsymbol{u}}) \, \mathrm{d}t \qquad \forall\, \widehat{\boldsymbol{u}} \in \mathcal{U},$$
$$\boldsymbol{w}(\cdot, T) = \dot{\boldsymbol{w}}(\cdot, T) = \boldsymbol{0} \qquad \text{in } \Omega.$$

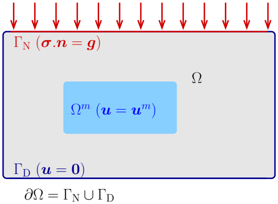
- Unique solvability for $(\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{U} \times \mathcal{W}$ expected if (i) sufficient data $\boldsymbol{u}_{\mathsf{obs}}$, (ii) cnds. on $\mathcal{D}_T$. (by analogy with time-harmonic elastic case [Aquino, B 19])

Some remarks

- If more-general boundary decomposition $\Gamma = \Gamma_{\mathsf{D}} \cup \Gamma_{\mathsf{N}} \cup \Gamma_c$ (with possibly $|\Gamma_c| \neq 0$), use
  $$\mathcal{U} := \big\{ \boldsymbol{v} \in \mathcal{V},\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_{\mathsf{D}} \big\}, \qquad \mathcal{W} := \big\{ \boldsymbol{v} \in \mathcal{V},\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_{\mathsf{D}} \cup \Gamma_c \big\} \subset \mathcal{U}.$$
- For well-posed BCs, $|\Gamma_c| = 0$ and $\mathcal{W} = \mathcal{U}$.



partially-specified BCs



well-posed BCs

- If additional prescribed excitations, weak balance of linear momentum becomes
  $$\int_\Omega \int_0^T \big(\boldsymbol{\sigma} : \boldsymbol{\varepsilon}[\boldsymbol{w}] + \rho \ddot{\mathbf{u}} \cdot \boldsymbol{w} \big)\, \mathrm{d}t\, \mathrm{d}V = \mathcal{F}(\boldsymbol{w}) \qquad \forall\, \boldsymbol{w} \in \mathcal{W},$$
- Initial (final) rest assumed for simplicity for $\boldsymbol{u}$ ($\boldsymbol{w}$). How to adapt to cases with unknown initial values for $\boldsymbol{u}$?
- Weighted ECR of form $\mathcal{E} = \mathcal{E}^{\mathrm{e}} + \gamma \mathcal{E}^{\mathrm{v}}$ easy to implement (e.g. stronger focus on dissipation).

MECR formulation for the time-harmonic case

- Conventions: $T = 2\pi/\omega$ and

$$\int_\Omega \int_0^T \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}V\,\mathrm{d}t \;=\; \frac{\pi}{\omega}\int_\Omega \mathrm{Re}(\boldsymbol{u}\cdot\overline{\boldsymbol{v}})\,\mathrm{d}V, \qquad \big\langle \partial_{\boldsymbol{x}} f, \widehat{\boldsymbol{x}} \big\rangle = \mathrm{Re}\big\{ (\partial_{\boldsymbol{x}_\mathrm{R}} f + \mathrm{i}\partial_{\boldsymbol{x}_\mathrm{I}} f)\cdot\overline{\widehat{\boldsymbol{x}}} \big\}$$

- Quadratic potentials become Hermitian forms, and

$$\psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) = \frac{1}{2}\big(\boldsymbol{\varepsilon}:\boldsymbol{\mathcal{C}}_\varepsilon:\bar{\boldsymbol{\varepsilon}} + \boldsymbol{\alpha}:\boldsymbol{\mathcal{C}}_\mathrm{m}:\bar{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}:\boldsymbol{\mathcal{C}}_\mathrm{m}:\bar{\boldsymbol{\alpha}} + \boldsymbol{\alpha}:\boldsymbol{\mathcal{C}}_\alpha:\bar{\boldsymbol{\alpha}}\big),$$

$$\boldsymbol{\sigma}^\mathrm{e}[\boldsymbol{u}] = \boldsymbol{\mathcal{C}}_\varepsilon:\boldsymbol{\varepsilon} + \boldsymbol{\mathcal{C}}_\mathrm{m}:\boldsymbol{\alpha}, \qquad \boldsymbol{\sigma}^\mathrm{v}[\boldsymbol{u}] = -\mathrm{i}\omega(\boldsymbol{\mathcal{D}}_\varepsilon:\boldsymbol{\varepsilon} + \boldsymbol{\mathcal{D}}_\mathrm{m}:\boldsymbol{\alpha}),$$

$$\varphi(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}}) = \omega^2\varphi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}), \qquad \partial_{\dot{\alpha}}\varphi(\dot{\boldsymbol{\varepsilon}},\dot{\boldsymbol{\alpha}}) = -\mathrm{i}\omega\partial_\alpha\varphi(\boldsymbol{\varepsilon},\boldsymbol{\alpha})$$

- Time-harmonic constitutive relation

$$\boldsymbol{\sigma}[\boldsymbol{u}] = (\boldsymbol{\sigma}^\mathrm{e} + \boldsymbol{\sigma}^\mathrm{v})[\boldsymbol{u}] = \boldsymbol{\mathcal{C}}(\omega):\boldsymbol{\varepsilon},$$

$$\boldsymbol{\mathcal{C}}(\omega) = (\boldsymbol{\mathcal{C}}_\varepsilon - \mathrm{i}\omega\boldsymbol{\mathcal{D}}_\varepsilon) - (\boldsymbol{\mathcal{C}}_\mathrm{m} - \mathrm{i}\omega\boldsymbol{\mathcal{D}}_\mathrm{m}):(\boldsymbol{\mathcal{C}}_\alpha - \mathrm{i}\omega\boldsymbol{\mathcal{D}}_\alpha)^{-1}:(\boldsymbol{\mathcal{C}}_\mathrm{m} - \mathrm{i}\omega\boldsymbol{\mathcal{D}}_\mathrm{m})$$

Stationarity system for the time-harmonic case

- Time-harmonic ECR functional:

$$\mathcal{E}(\mathbf{X},\mathbf{p};\omega) := \int_\Omega \Big\{ \epsilon_\psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha},\boldsymbol{\sigma}^{\mathrm{e}},\boldsymbol{A},\mathbf{p}) + T\epsilon_\varphi(\boldsymbol{\varepsilon},\boldsymbol{\alpha},\boldsymbol{\sigma}^{\mathrm{v}},\boldsymbol{A},\mathbf{p};\omega) \Big\} \,\mathrm{d}V,$$

  with Legendre-Fenchel gaps given by

$$\epsilon_\psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha},\boldsymbol{\sigma}^{\mathrm{e}},\boldsymbol{A},\mathbf{p}) := \psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha},\mathbf{p}) + \psi^\star(\boldsymbol{\sigma}^{\mathrm{e}},\boldsymbol{A},\mathbf{p}) - \mathrm{Re}\big[\boldsymbol{\sigma}^{\mathrm{e}}:\bar{\boldsymbol{\varepsilon}} - \boldsymbol{A}:\bar{\boldsymbol{\alpha}}\big],$$

$$\epsilon_\varphi(\boldsymbol{\varepsilon},\boldsymbol{\alpha},\boldsymbol{\sigma}^{\mathrm{v}},\boldsymbol{A},\mathbf{p};\omega) := \omega^2\varphi(\boldsymbol{\varepsilon},\boldsymbol{\alpha},\mathbf{p}) + \varphi^\star(\boldsymbol{\sigma}^{\mathrm{v}},\boldsymbol{A},\mathbf{p}) - \mathrm{Re}\big[\mathrm{i}\omega(\boldsymbol{\sigma}^{\mathrm{v}}:\bar{\boldsymbol{\varepsilon}} + \boldsymbol{A}:\bar{\boldsymbol{\alpha}})\big],$$

- MECR functional:

$$\Lambda_\kappa(\mathbf{X},\mathbf{p};\omega) := \mathcal{E}(\mathbf{X},\mathbf{p};\omega) + \tfrac{1}{2}\kappa\mathcal{D}(\boldsymbol{u}-\boldsymbol{u}_{\mathsf{obs}},\overline{\boldsymbol{u}-\boldsymbol{u}_{\mathsf{obs}}}),$$

- Lagrangian (with Lagrange multiplier $\boldsymbol{w}\in\mathcal{W}$):

$$\mathcal{L}(\mathbf{X},\boldsymbol{w},\mathbf{p};\omega) := \Lambda_\kappa(\mathbf{X},\mathbf{p};\omega) - \mathrm{Re}\Big\{ \int_\Omega \big((\boldsymbol{\sigma}^{\mathrm{e}}+\boldsymbol{\sigma}^{\mathrm{v}}):\boldsymbol{\varepsilon}[\bar{\boldsymbol{w}}] - \rho\omega^2\boldsymbol{u}\cdot\bar{\boldsymbol{w}}\big)\,\mathrm{d}V \Big\}.$$

  Local stationarity equations yield

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}}(\omega):\boldsymbol{\varepsilon} + \boldsymbol{\mathcal{S}}(\omega):\boldsymbol{\eta},$$

  where $\boldsymbol{\mathcal{S}}(\omega)$ Hermitian positive definite:

$$\boldsymbol{\mathcal{S}}(\omega) = \boldsymbol{\mathcal{C}}^{\mathsf{S}}_\varepsilon + \tfrac{1}{T}\boldsymbol{\mathcal{D}}_{\mathsf{I}} + \widehat{\boldsymbol{\mathcal{C}}}^{\mathsf{T}}:(\boldsymbol{\mathcal{C}}_\alpha - \mathrm{i}\omega\boldsymbol{\mathcal{D}}_\alpha)^{-1}:(\tfrac{1}{T}\boldsymbol{\mathcal{D}}_\alpha + \omega^2\boldsymbol{\mathcal{D}}_\alpha:\boldsymbol{\mathcal{C}}_\alpha^{-1}:\boldsymbol{\mathcal{D}}_\alpha):(\boldsymbol{\mathcal{C}}_\alpha + \mathrm{i}\omega\boldsymbol{\mathcal{D}}_\alpha)^{-1}:\widehat{\boldsymbol{\mathcal{C}}}.$$

Stationarity system for the time-harmonic case

- **Global stationarity equations:** A priori they read

$$\int_\Omega \big(\boldsymbol{\sigma} : \boldsymbol{\varepsilon}[\overline{\widehat{\boldsymbol{w}}}] - \rho\omega^2 \boldsymbol{u} \cdot \overline{\widehat{\boldsymbol{w}}}\big) \, \mathrm{d}V = 0 \qquad \forall\, \widehat{\boldsymbol{w}} \in \mathcal{W}.$$

$$\int_\Omega \Big\{ \big(\partial_\varepsilon \psi - \boldsymbol{\sigma}^{\mathrm{e}} - \mathrm{i}\omega T(\mathrm{i}\omega \partial_\varepsilon \varphi + \boldsymbol{\sigma}^{\mathrm{v}})\big) : \boldsymbol{\varepsilon}[\overline{\widehat{\boldsymbol{u}}}] + \rho\omega^2 \boldsymbol{w} \cdot \overline{\widehat{\boldsymbol{u}}} \Big\} \, \mathrm{d}V = -\kappa \mathcal{D}(\boldsymbol{u} - \boldsymbol{u}_{\mathsf{obs}}, \overline{\widehat{\boldsymbol{u}}}) \qquad \forall\, \widehat{\boldsymbol{u}} \in \mathcal{U}.$$

- Use results from local stationarity, to obtain

$$\int_\Omega \Big\{ \big(\boldsymbol{\mathcal{C}}(\omega) : \boldsymbol{\varepsilon}[\boldsymbol{u}]\big) : \boldsymbol{\varepsilon}[\overline{\widehat{\boldsymbol{w}}}] - \rho\omega^2 \boldsymbol{u} \cdot \overline{\widehat{\boldsymbol{w}}} \Big\} \, \mathrm{d}V + \int_\Omega \big(\boldsymbol{\mathcal{S}}(\omega) : \boldsymbol{\varepsilon}[\boldsymbol{w}]\big) : \boldsymbol{\varepsilon}[\overline{\widehat{\boldsymbol{w}}}] \, \mathrm{d}V = 0 \qquad \forall\, \widehat{\boldsymbol{w}} \in \mathcal{W}$$

$$\int_\Omega \Big(\big(\overline{\boldsymbol{\mathcal{C}}(\omega)} : \boldsymbol{\varepsilon}[\boldsymbol{w}]\big) : \boldsymbol{\varepsilon}[\overline{\widehat{\boldsymbol{u}}}] - \rho\omega^2 \boldsymbol{w} \cdot \overline{\widehat{\boldsymbol{u}}} \Big) \, \mathrm{d}V - \kappa \mathcal{D}(\boldsymbol{u} - \boldsymbol{u}_{\mathsf{obs}}, \overline{\widehat{\boldsymbol{u}}}) = 0 \qquad \forall\, \widehat{\boldsymbol{u}} \in \mathcal{U}$$

- Stationarity system: perturbed mixed pb. (as in [Aquino, B 19] for time-harmonic elasticity)

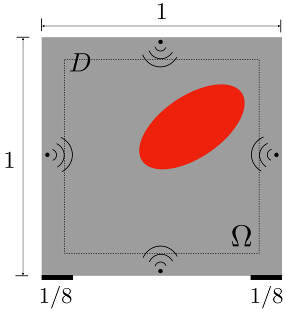
$$\begin{aligned} &\mathsf{S}(\boldsymbol{w}, \widehat{\boldsymbol{w}}; \omega) + \mathsf{C}(\boldsymbol{u}, \widehat{\boldsymbol{w}}; \omega) = 0 && \forall\, \widehat{\boldsymbol{w}} \in \mathcal{W}, \\ &\overline{\mathsf{C}}(\boldsymbol{u}, \widehat{\boldsymbol{u}}; \omega) - \kappa \mathcal{D}(\boldsymbol{u}, \widehat{\boldsymbol{u}}) = -\kappa \mathcal{D}(\boldsymbol{u}_{\mathsf{obs}}, \overline{\widehat{\boldsymbol{u}}}) && \forall\, \widehat{\boldsymbol{u}} \in \mathcal{U}, \end{aligned}$$

$$\mathsf{C}(\boldsymbol{u}, \widehat{\boldsymbol{w}}; \omega) = \big(\boldsymbol{\mathcal{C}}(\omega) : \boldsymbol{\varepsilon}[\boldsymbol{u}],\, \boldsymbol{\varepsilon}[\widehat{\boldsymbol{w}}]\big)_\Omega - \omega^2 \big(\rho \boldsymbol{u},\, \widehat{\boldsymbol{w}}\big)_\Omega,$$

$$\mathsf{S}(\boldsymbol{w}, \widehat{\boldsymbol{u}}; \omega) = \big(\boldsymbol{\mathcal{S}}(\omega) : \boldsymbol{\varepsilon}[\boldsymbol{w}],\, \boldsymbol{\varepsilon}[\widehat{\boldsymbol{w}}]\big)_\Omega$$

- Key property: S coercive on $\mathcal{W} \times \mathcal{W}$.

Numerical results (computations by P. Salasiya, B. Guzina)



- (possibly overlapping) subzones $\Omega = S_1 \cup \ldots \cup S_M$
- Identify heterogeneous $\mathbf{p}$ subzone-wise
- Computational experiments: 4 sources, 2 excitation directions each, 4 frequencies, $M = 4 \times 4$ square subzones
- Synthetic data generated with fine mesh ($h = .0025$, $p = 3$);
- Identification, stationarity solves etc. performed with coarser mesh ($h = .05$, $p = 3$), avoids "inverse crime";
- $\widetilde{\Lambda}_\kappa(\mathbf{p})$ minimized using SLSQP algorithm, $\mathbf{p}$ pixel-wise constant, $\mathcal{N} \times \mathcal{N}$ pizels per subzone, resolution refinement $\mathcal{N} = 1, 5, 7$.
- Noisy data: under progress, κ set by seeking the "corner" of ther L-curve.

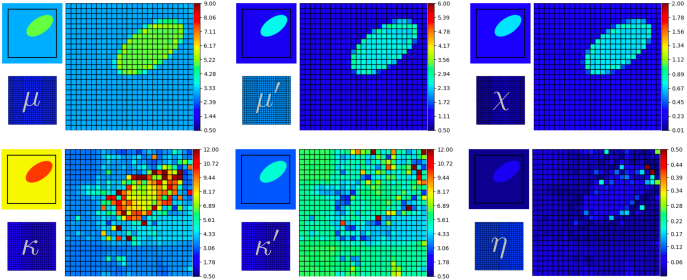
following Tan et al. (2016), McGarry et al. (2022) for elastography

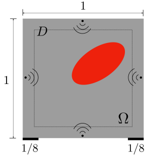
Numerical results (P. Salasiya, B. Guzina)

Isotropic standard linear solid model, $\mathbf{p} = (\kappa, \mu, \kappa', \mu', \eta, \chi)$

$$\boldsymbol{\mathcal{C}}(\mathbf{p}, \omega) = \frac{3\kappa(\kappa' - \mathrm{i}\omega\eta)}{\kappa + \kappa' - \mathrm{i}\omega\eta}\boldsymbol{\mathcal{J}} + \frac{2\mu(\mu' - \mathrm{i}\omega\chi)}{\mu + \mu' - \mathrm{i}\omega\chi}\boldsymbol{\mathcal{K}},$$

Parameter	κ	μ	κ'	μ'	η	χ
Background	8	3	3	1	0.01	0.3
Inclusion	10	5	5	2.5	0.05	0.7
Initial guess	2	2	2	2	0.005	0.005



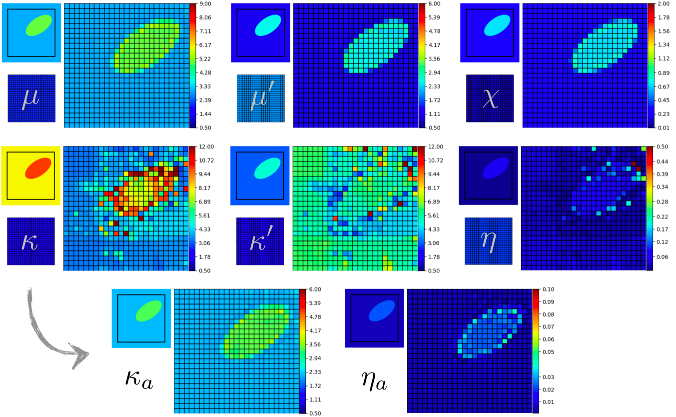


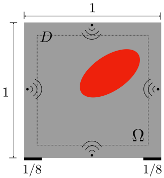
Numerical results (P. Salasiya, B. Guzina)

Parameter	κ	μ	κ'	μ'	η	χ
Background	8	3	3	1	0.01	0.3
Inclusion	10	5	5	2.5	0.05	0.7
Initial guess	2	2	2	2	0.005	0.005

Kelvin-Voigt bulk approximation (due to $\eta \ll 1$):

$$K(\mathbf{p},\omega) = \kappa_a - \mathrm{i}\eta_a\omega + O((\eta\omega)^2), \qquad \kappa_a = \frac{\kappa\kappa'}{\kappa+\kappa'}, \quad \eta_a = \frac{\kappa^2\eta}{(\kappa+\kappa')^2}.$$





- Stationarity problem for the transient case:
  - ▷ well-posedness results, conditions on the data?
  - ▷ potential computational bottleneck

Thank you for your kind attention!
Any questions?