Tsunami identification from surface data: an illustration of the Tikhonov-Morozov strategy

Laurent Bourgeois

Laboratoire POEMS, ENSTA Paris, CNRS/ENSTA/INRIA
Palaiseau, France

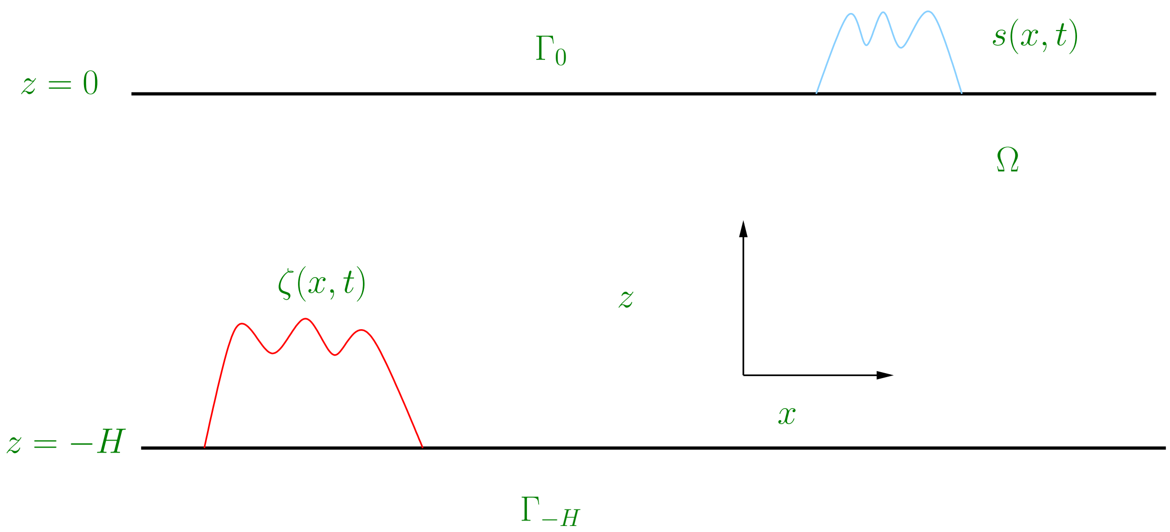
PhD of Raphaël Terrine

Quiberon, Ecole de Mécanique

September 2024

A simple ocean model (time domain)

Velocity potential $\varphi(x,z,t)$, free surface perturbation $s(x,t)$, vertical bottom displacement $\zeta(x,t)$

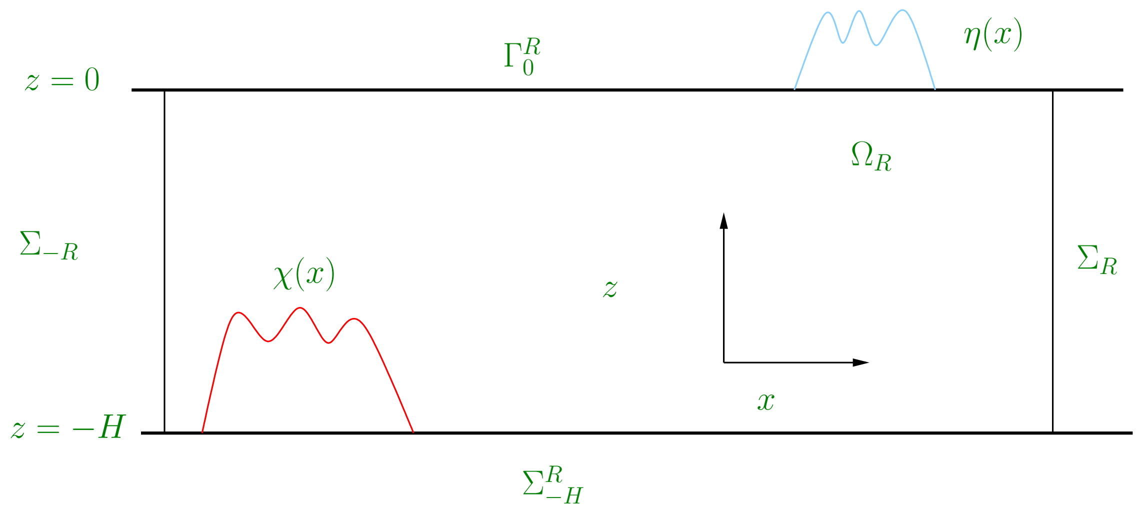


Potential φ satisfies:

$$\begin{cases} (1/c^2)\partial_t^2\varphi - \Delta\varphi = 0 & \text{in } \Omega \times (0,+\infty) \\ \partial_t\varphi + g\,s = 0 & \text{on } \Gamma_0 \times (0,+\infty) \\ \partial_z\varphi - \partial_t s = 0 & \text{on } \Gamma_0 \times (0,+\infty) \\ \partial_z\varphi = \partial_t\zeta & \text{on } \Gamma_{-H} \times (0,+\infty) \end{cases}$$

The frequency domain: $\varphi(x,z,t) = u(x,z)e^{-i\omega t}$

Velocity potential $u(x,z)$, free surface perturbation $\eta(x)$, vertical bottom displacement $\chi(x)$



Potential u satisfies ($T_\pm$: Dirichlet-to-Neumann operators):

$$\left\{\begin{array}{rcll} \Delta u + (\omega^2/c^2)u & = & 0 & \text{in } \Omega^R \\ \partial_z u - (\omega^2/g)u & = & 0 & \text{on } \Gamma_0^R \\ \partial_z u & = & -i\omega\chi & \text{on } \Gamma_{-H}^R \\ \pm\partial_x u - T_\pm u & = & 0 & \text{on } \Sigma_{\pm R} \end{array}\right.$$

The tsunami inverse problem

From the measurement of the **free surface** $\eta = (i\omega/g)u$ on Γ_0, find the **tsunami** χ on Γ_{-H} $\longrightarrow$ equivalent to a **strongly ill-posed** Cauchy problem for the Helmholtz equation

Compute u such that:

$$\begin{cases} \Delta u + (\omega^2/c^2)u = 0 & \text{in } \Omega^R \\ \partial_z u - (\omega^2/g)u = 0 & \text{on } \Gamma_0^R \\ u = u^{\text{mes}} & \text{on } \Gamma_0^R \\ \pm\partial_x u - T_\pm u = 0 & \text{on } \Sigma_{\pm R} \end{cases}$$

with $u^{\text{mes}} := (g/i\omega)\eta^{\text{mes}}$

then compute $\chi = (i/\omega)\partial_z u$ on Γ^R_{-H}

The tsunami inverse problem (cont.)

An abstract framework

Spaces:

$$V = H^1(\Omega^R), \quad M = \{\mu \in H^1(\Omega^R),\, \mu|_{\Gamma^R_{-H}} = 0\}, \quad O = L^2(\Gamma^R_0)$$

Operators: $B : V \to M$ defined by

$$\begin{aligned} b(v,\mu) &= \int_{\Omega^R} \left(\nabla v \cdot \nabla \overline{\mu} - (\omega^2/c^2) v \overline{\mu}\right) dxdz - (\omega^2/g) \int_{\Gamma^R_0} v\, \overline{\mu}\, dx \\ &- \langle T_+ v, \overline{\mu} \rangle_{H^{-1/2}(\Sigma_R), \tilde{H}^{1/2}(\Sigma_R)} - \langle T_- v, \overline{\mu} \rangle_{H^{-1/2}(\Sigma_{-R}), \tilde{H}^{1/2}(\Sigma_{-R})} \\ &= (Bv, \mu)_M \end{aligned}$$

$C : V \to O$ is the trace operator,

$A : V \to M \times O$ is defined as $A = (B, C)$

Data: is given by $F = (0, u^{\text{mes}}) \in M \times O$

The tsunami inverse problem

The inverse problem: for $F \in M \times O$, find $u \in V$ such that $Au = F$ (A is injective, dense range but not onto) $\longrightarrow$ we apply the Tikhonov-Morozov strategy for A

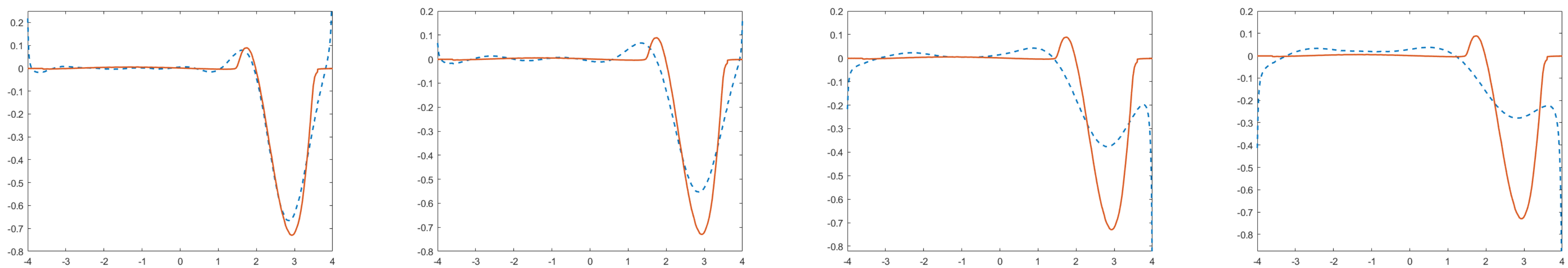
Mixed Tikhonov formulation: find $(u_\varepsilon, \lambda_\varepsilon) \in V \times M$ such that

$$\left\{ \begin{array}{c} \varepsilon(u_\varepsilon, v)_{H^1(\Omega^R)} + \displaystyle\int_{\Omega^R} \nabla\lambda_\varepsilon \cdot \nabla\overline{v} - (\omega^2/c^2)\lambda_\varepsilon \overline{v}\, dxdz - (\omega^2/g)\int_{\Gamma_0^R} \lambda_\varepsilon\, \overline{v}\, dx \\ -\langle \overline{T_+}\lambda_\varepsilon, \overline{v}\rangle_{H^{-1/2}(\Sigma_R),\tilde{H}^{1/2}(\Sigma_R)} - \langle \overline{T_-}\lambda_\varepsilon, \overline{v}\rangle_{H^{-1/2}(\Sigma_{-R}),\tilde{H}^{1/2}(\Sigma_{-R})} \\ + \displaystyle\int_{\Gamma_0^R} u_\varepsilon\, \overline{v}\, dx = \int_{\Gamma_0^R} u^{\text{mes}}\, \overline{v}\, dx, \qquad \forall v \in V \\ \displaystyle\int_{\Omega^R} \nabla u_\varepsilon \cdot \nabla\overline{\mu} - (\omega^2/c^2) u_\varepsilon \overline{\mu}\, dxdz - (\omega^2/g)\int_{\Gamma_0^R} u_\varepsilon\, \overline{\mu}\, dx \\ -\langle T_+ u_\varepsilon, \overline{\mu}\rangle_{H^{-1/2}(\Sigma_R),\tilde{H}^{1/2}(\Sigma_R)} - \langle T_- u_\varepsilon, \overline{\mu}\rangle_{H^{-1/2}(\Sigma_{-R}),\tilde{H}^{1/2}(\Sigma_{-R})} \\ - \displaystyle\int_{\Omega^R} \nabla\lambda_\varepsilon \cdot \nabla\overline{\mu}\, dxdz = 0, \qquad \forall \mu \in M \end{array} \right.$$

Numerical experiments

Artificial data obtained with the **complete model**

Inverse problem solved with the **gravity model** (case $c \to +\infty$) or the **acoustic model** (case $g \to 0$).



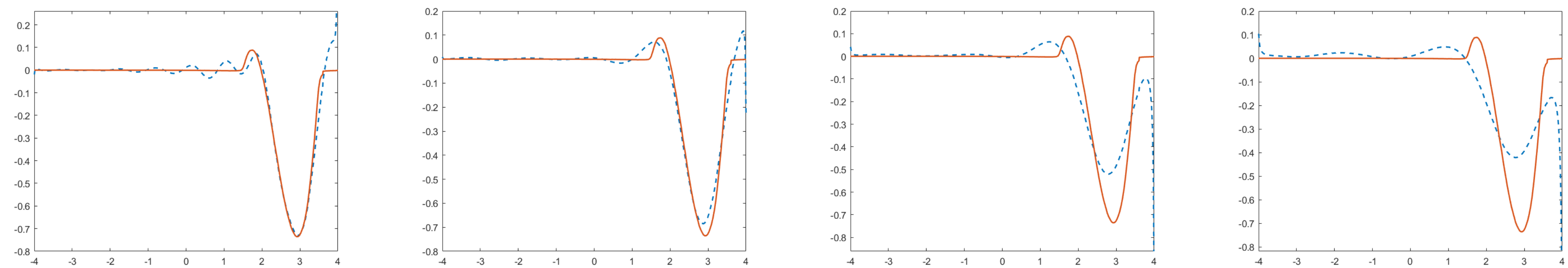
Exact data, 1% noise, 5% noise, 10% noise

Gravity case for $\omega = 3$, comparison between retrieved χ (dashed line) and exact χ (continuous line) for various amplitudes of noise

Numerical experiments (cont.)

Artificial data obtained with the **complete model**

Inverse problem solved with the **gravity model** (case $c \to +\infty$) or the **acoustic model** (case $g \to 0$).

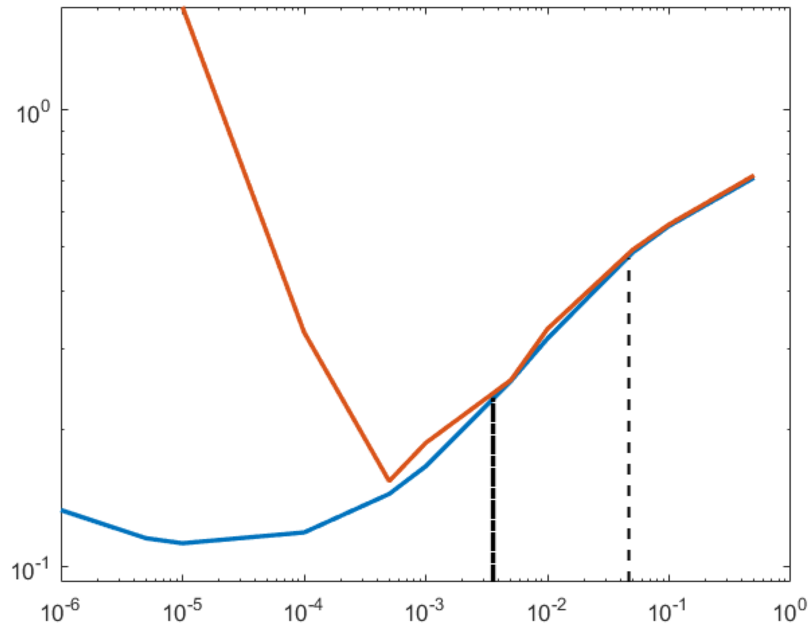


Exact data, 1% noise, 5% noise, 10% noise

Acoustic case for $\omega = 20$, comparison between retrieved χ (dashed line) and exact χ (continuous line) for various amplitudes of noise

Numerical experiments (cont.)

How good is the Morozov choice ?



Error $\|\chi_\varepsilon^\delta - \chi\|_{L^2(\Gamma_0^R)}$ in the acoustic case as a function of ε

Blue: no noise, **Red**: 5% noise.

The vertical dashed thin/thick lines corresponds to the Morozov choice following deterministic/probabilistic procedure to lift Neumann data