Ill Posed Problems

Faker, Ben Belgacem

LMAC-UTCompiègne

QUIBERON Sept. 2024

Ill-Posedness?

Intuitive definition?

- It is all about solving mathematical problems (Algebraic Equations (AE), Ordinary Differential Equations (ODE), Algebraic Differential Equations (ADE), Partial Differential Equations (PDE), . . .).
- Ill-posed problems are *hard to solve* — unless everything is perfect (impossible!).

Mathematical definition?

- Hadamard tryptic for well posedness : Existence-Uniqueness-Stability.
- A non-well posed problem is an ill posed problem ($\Longrightarrow$) At least one of the three rings E.-U.-S. is missing.

Abstract examples

Set $I = (0, \pi)$. Given the kernel $K \in L^2(I \times I)$.
Define the Fredholm operator

$$A : L^2(I) \to L^2(I), \qquad f \mapsto Af(s) = \int_I K(s,t) f(t)\, dt.$$

LEM. **1** *A is a compact operator* ($\Longrightarrow$) *A^{-1} cannot be continuous.*

PROOF : Hilbert Schmidt theorem ($\Longrightarrow$) The singular values $(\sigma_k)_{k \geq 0}$ of A decay toward zero. A^{-1} cannot be bounded.

THE PROBLEM : FIND $f \in L^2(I)$ SUCH THAT

$$Af = g$$

IS ILL POSED, BECAUSE OF THE INSTABILITY.

Ill-posedness Degree?

Terminology by G. Wahba 1980.

Def. **2 (B. Hoffmann)** *The compactness degree of A is related to the 'decreasing rate' of the singular values sequence $(\sigma_k)_{k\geq 0}$. It is defined to be the real number*

$$q = \lim_{k\to\infty} -\frac{\ln(\sigma_k)}{\ln k}$$

Def. **3** *If A is compact, the ill posedness degree of problem $Af = g$ is the comptactness degree of A.*

Rem. 1 — *Mild ill posedness* $(0 < q < 1)$
— *Moderate ill posedness* $(1 < q < \infty)$
— *Severe ill posedness* $(q = \infty)$

Smoothness of the kernel

Expand the kernel $K(s,t)$ on the Legendre polynomials

$$K(s,t) = \underbrace{\sum_{0\leq j\leq p-1} a_j(s)L_j(t)}_{K_k(s,t)} + E_k(s,t)$$

A_p is the operator defined by means of K_p, Rank $A_p = p$.

PROP. **4** *If $K \in L^2(I, H^m(I))$ with $m \geq 1$. Then, the compactness degree of A is at least m.*

PROOF : Observe first that

$$\|A - A_p\|_{\mathcal{L}(L^2(I),L^2(I))} \leq \|E_p\|_{L^2(I\times I)} \leq Cp^{-m}.$$

Then,

$$\sigma_p \leq \inf_{Rank\ D_p=p} \|A - D_p\|_{\mathcal{L}(L^2(I),L^2(I))} \leq \|A - A_p\|_{\mathcal{L}(L^2(I),L^2(I))}.$$

Inverse Heat transfer problems

Initial State Reconstruction

Heat Equation

Let $I = (0, \pi)$ and $Q = I \times]0, T[$. The heat equation

$$
\begin{aligned}
\partial_t y - \partial_{xx} y &= 0 && \text{in } Q, \\
y(0, \cdot) = 0, \qquad y(\pi, \cdot) &= 0 && \text{on } (0, T), \\
y(\cdot, 0) &= \varphi && \text{on } I.
\end{aligned}
$$

Direct problem:
The initial state φ is known, find the whole temperature field $y(t, (\cdot))$.

Inverse problem:
The final state $y(T, (\cdot))$ is known (observed), reconstruct φ.

Fourier Method

Hilbert (Fourier) basis in $L^2(I)$

$$e_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx), \qquad -\partial_{xx}e_k(x) = k^2 e_k(x), \qquad k \geq 1$$

Projection of the initial condition and the solution

$$\varphi(x) = \sum_{k\geq 1}\varphi_k e_k(x); \qquad y(t,x) = \sum_{k\geq 1} y_k(t)e_k(x)$$

Then, solve the ODE

$$y_k'(t) + k^2 y_k(t) = 0 \qquad \text{in } (0,T),$$
$$y_k(0) = \varphi_k$$

Easy computations produce

$$y(t,x) = \sum_{k\geq 1} y_k(t)e_k(x) = \sum_{k\geq 1}\varphi_k e^{-k^2 t}e_k(x)$$

Final State (at $t = T$)

Consider the operator

$$B : L^2(I) \to L^2(I)$$
$$\varphi \to y(\cdot, T) = \sum_{k \geq 1} \varphi_k e^{-k^2 T} e_k(x)$$

B is continuous

$$\|B\varphi\|_{L^2(I)} \leq \|\varphi\|_{L^2(I)}$$

Smoothness: $B\varphi$ is an analytic function in I.

Spectrum $(B) = \{e^{-k^2 T}, k \geq 1\}$ $(\Longrightarrow)$ B is a compact operator. The range $R(B)$ is not closed in $L^2(I)$.

COMPACTNESS DEGREE : EXPONENTIAL

Reconstruction

WE KNOWN NOTHING ABOUT φ.

Assume that $y(\cdot, T)$ is observed, $\psi = y(\cdot, T)(?)$ is the (inexact!) observation.
Reconstruction of φ

$$\psi = \sum_{k \geq 1} \psi_k e_k(x) \quad (\Longrightarrow) \quad \varphi = B^{-1}\psi = \sum_{k \geq 1} \psi_k e^{k^2 T} e_k(x)$$

QUESTION : $\varphi \in L^2(I)$?

$$\|\varphi\|_{L^2(I)} = \sqrt{\sum_{k \geq 1} (\psi_k)^2 e^{2k^2 T}} < \infty? \qquad \text{ANSWSER : NO!}$$

Spectrum $(B^{-1}) = \{e^{k^2 T}, k \geq 1\}$ $(\Longrightarrow)$ B^{-1}, an unbounded operator

Irreversiblity?

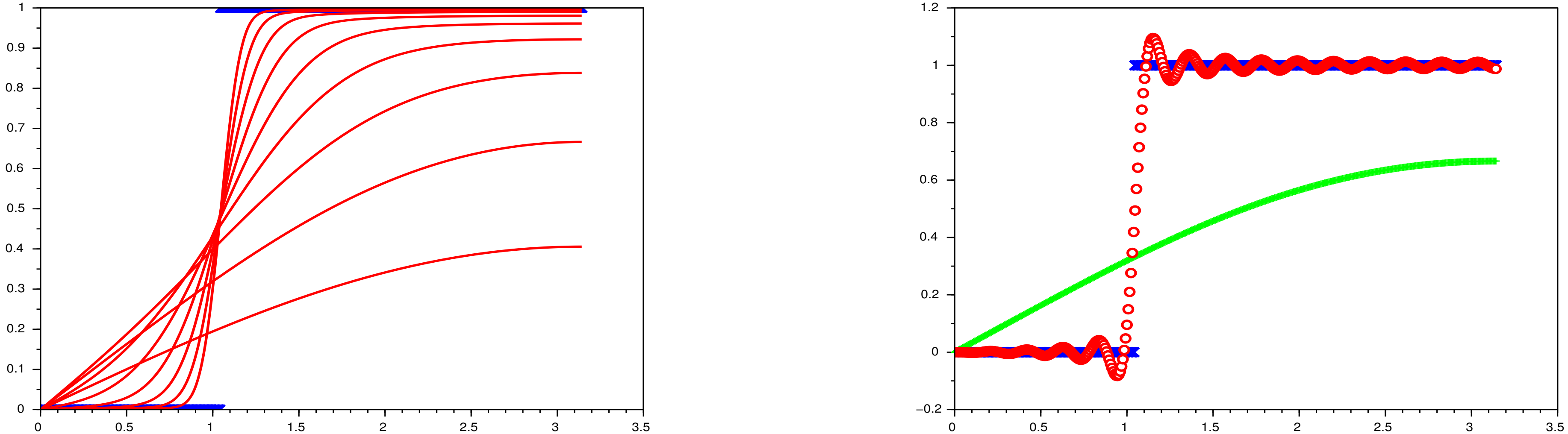
The heat transfer equation, with a final condition

$$\begin{aligned} \partial_t y - \partial_{xx} y &= 0 && \text{in } Q, \\ y(0,\cdot) = 0, \qquad y(\pi,\cdot) &= 0 && \text{on } (0,T), \\ y(T,0) &= \psi && \text{on } I. \end{aligned}$$

PROP. **5** *No solutions for so many ψ (proven previously).*

THAT'S WHY : WE CURRENTLY SAY THAT THE HEAT DIFFUSION IS AN IRREVERSIBLE PROCESS. OR, THE HEAT EQUATION IS NOT TIME-REVERSIBLE.

NUMERICS



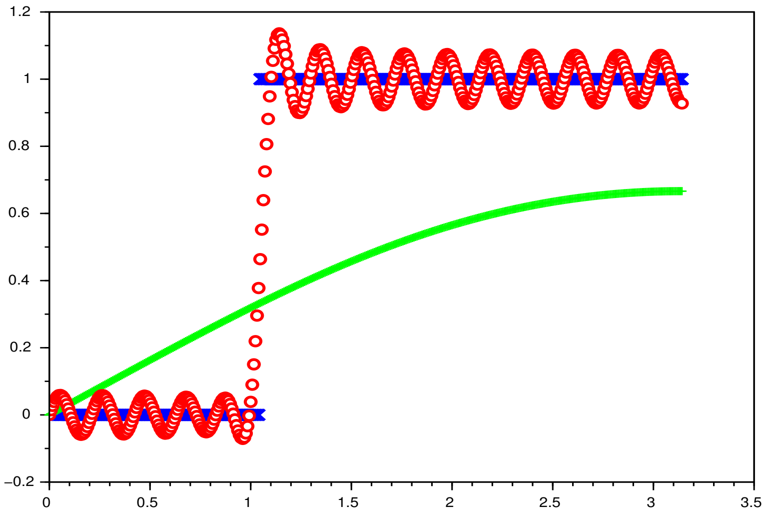
The temperature for $t(0 \leq t \leq 1/2)$ (left).

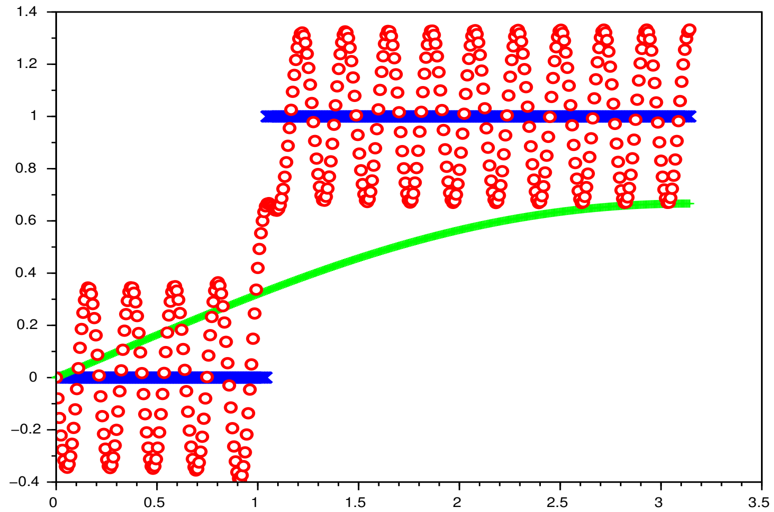
Reconstruction of the inital state from $T(1/2, (\cdot))$ ~~through Fourier coef.~~ (right)

White Noise on the observations: $T(1/2, (\cdot)) + \epsilon_k$

$$\epsilon_k \sim \mathcal{N}(0, \sigma^2), \quad \sigma =, k \geq 100$$

Numerics





White Noise

$$\epsilon_k \sim \mathcal{N}(0, \sigma^2), \qquad \sigma = 1 \times 10^{-k}$$

$$k = 95 \text{ (left)} \qquad k = 94 \text{ (right)}$$

A Controllability problem

Boundary Control operator

Let $u = u(t) \in L^2(0,T)$ be given.

The state heat equation

$$\begin{aligned} \partial_t y_u - \partial_{xx} y_u &= 0 \qquad \text{in } Q, \\ y_u(0,t) = 0, \qquad y_u(\pi,t) &= u(t) \qquad \text{on } (0,T), \\ y_u(x,0) &= 0 \qquad \text{on } I. \end{aligned}$$

The control operator

$$(Bu)(x) = y_u(x,T) \in L^2(I)?$$

Fourier Analysis (of B)

Modify the boundary condition so to use Fourier basis

$$y_u(t,x) = z(t,x) + \frac{x}{\pi} u(t)$$

Then

$$\begin{aligned} \partial_t z - \partial_{xx} z &= -\frac{x}{\pi} u'(t) \qquad && \text{in } Q, \\ z(0,t) = 0, \qquad z(\pi,t) &= 0 && \text{on } (0,T), \\ z(x,0) &= -\frac{x}{\pi} u(0) && \text{on } I. \end{aligned}$$

Fourier expansion of that RHS and IC

$$\frac{x}{\pi} = \sum_{k\geq 1} a_k e_x(x) = \sum_{k\geq 1} (-1)^{k+1} \frac{\pi}{k} e_x(x)$$

Formal Calculations

Fourier expansion of z

$$z(t,x) = \sum_{k\geq 1} z_k(t) e_x(x)$$

Then

$$\begin{aligned} z_k'(t) + k^2 z_k(t) &= -a_k u'(t) \qquad\qquad \text{in } (0,T), \\ z_k(0) &= -a_k u(0). \end{aligned}$$

Solve the equation (Details on the black board!)

$$z_k(t) = -a_k u(t) + (-1)^{k+1} k \int_0^t u(s) e^{-k^2(T-s)}\, ds$$

Back to the control operator

$$Bu = y_u(T,x) = z(T,x) + \frac{x}{\pi} u(T) = \sum_{k\geq 1} [z_k(T) + a_k u(T)] e_k(x)$$

Explicit form of B

The Control operator B is unbounded (non-continuous)

$$B : D(B) \subset L^2(0,T) \to L^2(I)$$

$$u \to Bu(x) = \sum_{k\geq 1}(-1)^{k+1}\left[k\int_0^T u(t)e^{-k^2(T-t)}\,dt\right]e_k(x)$$

The adjoint operator is easily derived through the identity

$$(Bu,\varphi)_{L^2(I)} = (u, B^*\varphi)_{L^2(0,T)}$$

It is expressed as

$$B^* : D(B^*) \subset L^2(I) \to L^2(0,T)$$

$$\varphi \to B^*\varphi(t) = \sum_{k\geq 1}(-1)^{k+1}k\,\varphi_k\,e^{-k^2(T-t)}$$

Properties of B

Density of the domains

$$\overline{\mathbb{D}(B)} = L^2(0,T), \qquad \overline{\mathbb{D}(B^*)} = L^2(I)$$

We have

$$\mathcal{N}(B^*) = \{0\} \left[(\Longrightarrow) \quad \overline{\mathcal{R}(B)} = L^2(I) \right]$$

A sharp analysis yields (L. Schwartz, Thesis 1937),

$$\dim \overline{\mathcal{R}}(B^*)^{\perp} = +\infty \quad (\Longrightarrow) \quad \dim \mathcal{N}(B) = +\infty$$

Caution! $^{\perp}$ is taken in $L^2(0,T)$.

Exact Controllability

Let $y_T \in L^2(I)$ a fixed (desired!) state.

Exact-controllability $(\Longrightarrow)$ *find* $u = u(t)$ *satisfying*

$$(Bu)(x) = y_u(x,T) = y_T(x)$$

The exact-controllability may FAIL $(\Longleftrightarrow)$ $\mathcal{R}(B) \neq L^2(I)$.

UNIQUENESS : **Fails!** dim $\mathcal{N}(B) \neq \{0\}$.

EXISTENCE AND STABILITY : Interconnected, through the open map theorem!

Control Space!

We have (the control space!) —(Clarkson-Erdös-Schwartz theorem)—

$$\overline{\mathcal{R}(B^*)} = \left\{ v \in L^2(0,T), \qquad v(t) = \sum_{k\geq 1} v_k e^{-k^2(T-t)} \right\}$$

Rem. 2 *We have*

$$v \in \overline{\mathcal{R}(B^*)} \quad (\Longleftrightarrow) \quad \|v\|^2_{L^2(I)} = \sum_{k\geq 1}\sum_{m\geq 1} \frac{1-e^{-(k^2+m^2)T}}{k^2+m^2} v_k v_m < \infty$$

$$v \in \mathcal{R}(B^*) \quad (\Longleftrightarrow) \quad \|\varphi\|^2_{L^2(I)} = \|(B^*)^{-1} v\|^2_{L^2(I)} = \sum_{k\geq 1} (\varphi_k)^2 = \sum_{k\geq 1} \left(\frac{v_k}{k}\right)^2 < \infty.$$

Non-Closeness of the ranges (Nasty! Source of ill-posedness, later)

$$\mathcal{R}(B) \neq \overline{\mathcal{R}(B)} = L^2(I), \qquad \mathcal{R}(B^*) \neq \overline{\mathcal{R}(B^*)}$$

HUM CONTROL

First, Fourier expansion of y_T

$$y_T(x) = \sum_{k\geq 1} (y_T)_k e_k(x).$$

The HUM control $u^\dagger \in \overline{\mathcal{R}(B^*)}$ that is ([a])

$$u^\dagger(t) = \sum_{m\geq 1} (u^\dagger)_m \, e^{-m^2(T-t)}, \qquad \forall t.$$

Plugging in the explicit expression of $Bu^\dagger$, yields that

$$(-1)^{k+1}k \sum_{m\geq 1} \frac{1-e^{-(k^2+m^2)T}}{k^2+m^2}(u^\dagger)_m = (y_T)_k, \qquad \forall k.$$

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[a] Uniqueness is restored! Owing to $\overline{\mathcal{R}(B^*)} = \mathcal{N}(B)^\perp$.

An infinite linear system

Define the infinite matrices $\mathcal{C}_T, \mathcal{D}$

$$\mathcal{C}_T = (c_{km})_{k,m} = \left(\frac{(1 - e^{-(k^2+m^2)T})}{k^2 + m^2} \right), \qquad \mathcal{D} = (d_{km})_{k,m} = \left((-1)^{k+1} \frac{\delta_{km}}{k} \right).$$

We derive that $(\boldsymbol{u}^\dagger = ((u^\dagger)_{m \geq 1}))$

$$\boldsymbol{u}^\dagger = (\mathcal{C}_T)^{-1} \mathcal{D}\, \boldsymbol{y}_T.$$

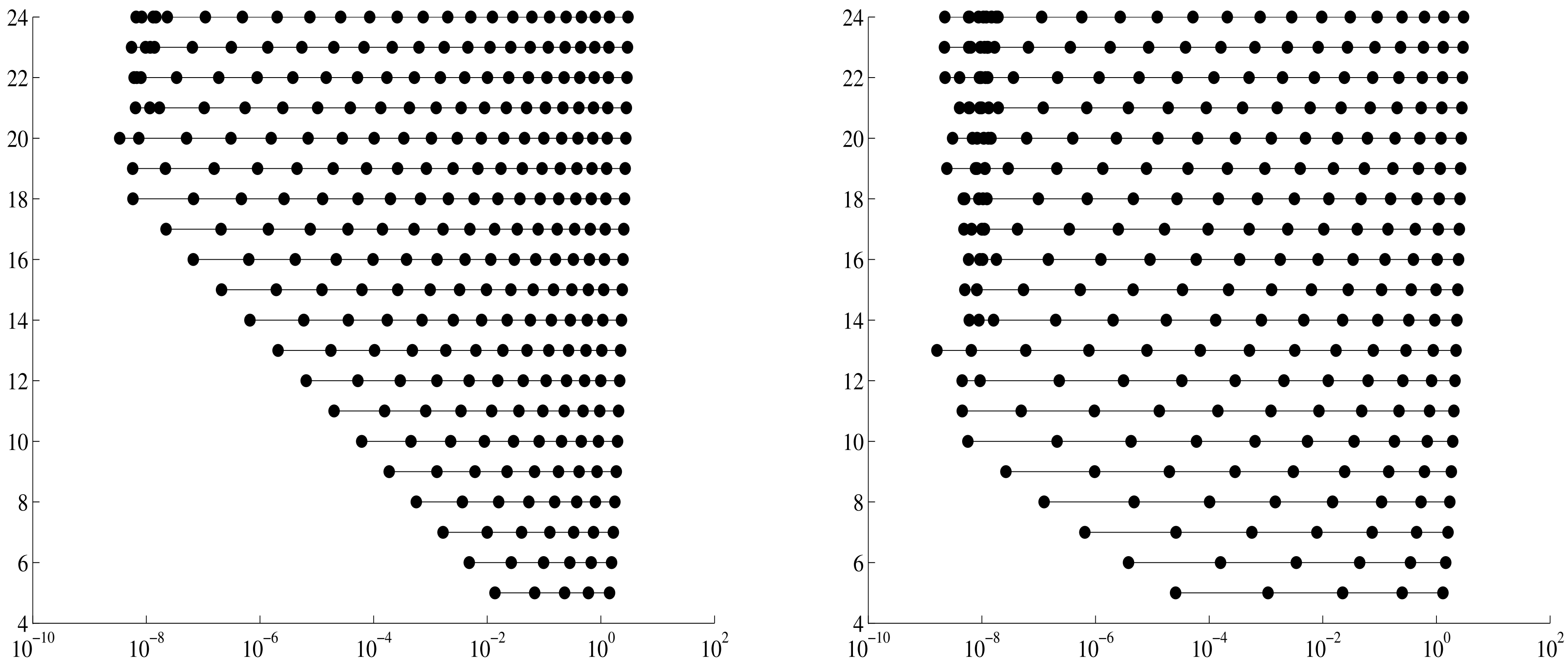
Condition for $\boldsymbol{u}^\dagger$ to be in $L^2(0,T)$ is

$$\|\boldsymbol{u}^\dagger\|^2_{L^2(0,T)} = \sum_{k \geq 1} \sum_{m \geq 1} \frac{1 - e^{-(k^2+m^2)T}}{k^2 + m^2} (u^\dagger)_m (u^\dagger)_k = (\mathcal{C}_T \boldsymbol{u}^\dagger,\, \boldsymbol{u}^\dagger)_{\ell^2(\mathbb{R})} < \infty,$$

or again

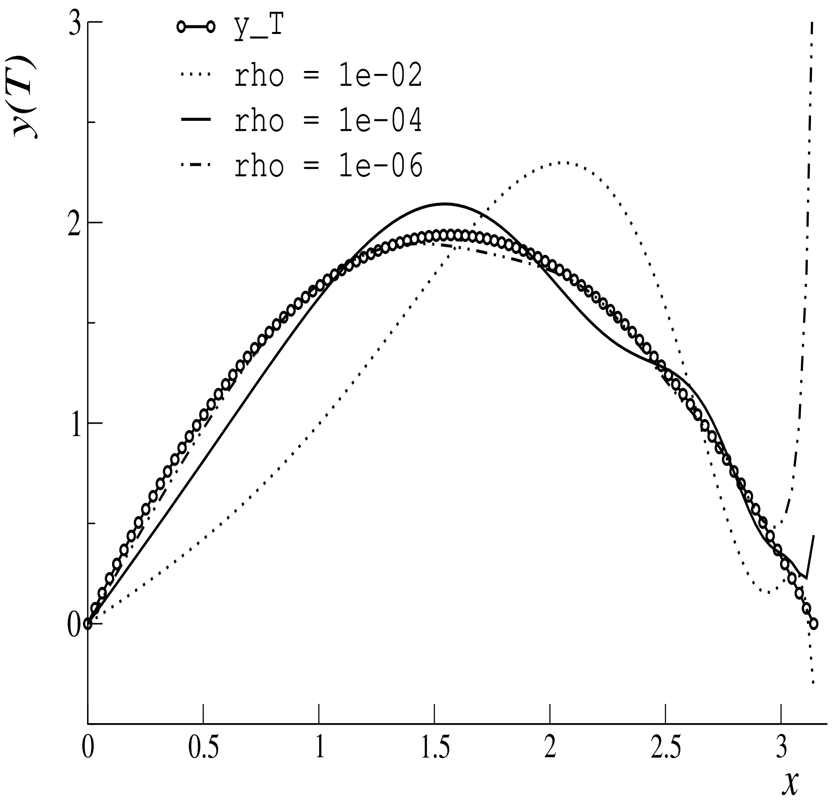
$$(\boldsymbol{y}_T,\, \mathcal{D}(\mathcal{C}_T)^{-1} \mathcal{D} \boldsymbol{y}_T)_{\ell^2(\mathbb{R})} < \infty.$$

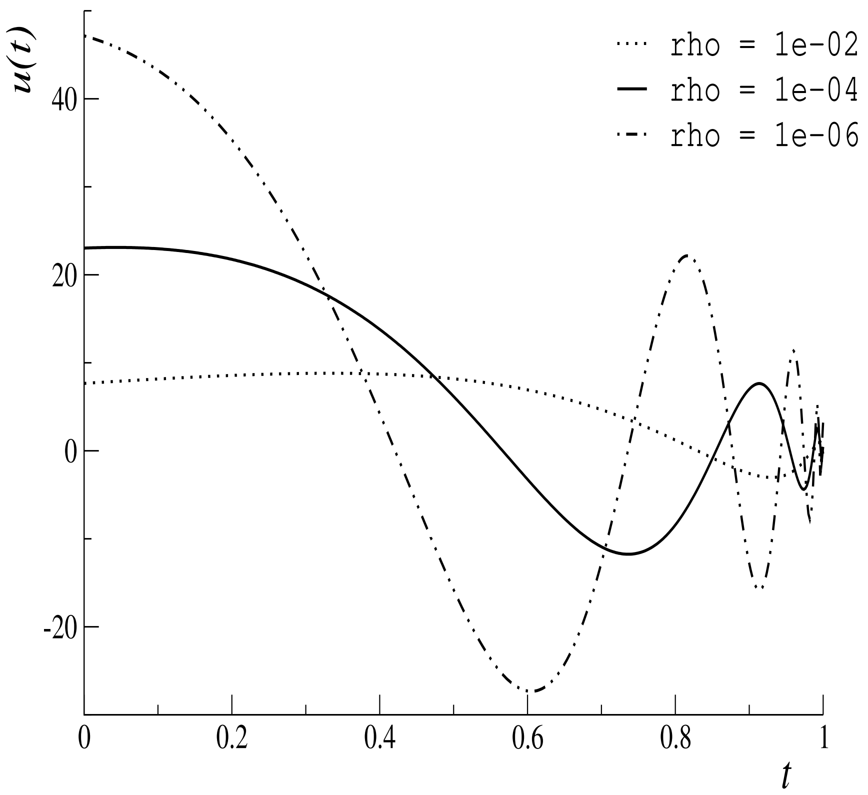
EIGENVALUES



Eigenvalues of $(\mathcal{C}_T)$ $T = 1$ (left) $T = 0.1$ (right).

Exact-Controllability





Computed Controlled states. The related controls ($T = 1$).