Data Completion Problem
Variational Formulations

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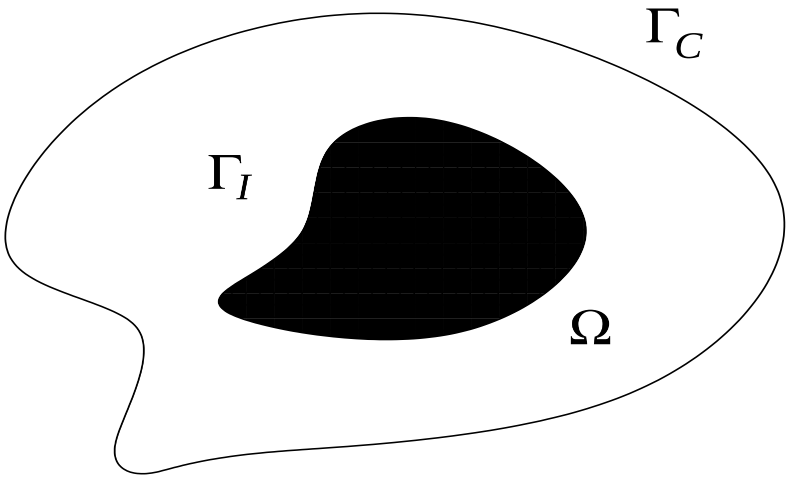
The Model

Γ_I : unreachable ($\Longrightarrow$) No measures

($\Longrightarrow$) No Boundary C.

Γ_C : acessible ($\Longrightarrow$) A lot of measures

($\Longrightarrow$) Two Boundary C.s



VERY SIMPLE MODEL. : *Find a potential u such that*

$$
\begin{array}{rcll}
-\operatorname{div}(a\nabla u) & = & 0 & \text{in } \Omega, \\
u & = & g & \text{on } \Gamma_C \quad \text{Dirichlet condition}, \\
a\partial_{\boldsymbol{n}} u & = & \varphi & \text{on } \Gamma_C. \quad \text{Neumann condition} \\
u & = & ? & \text{on } \Gamma_I.
\end{array}
$$

With volume source?

DEFINE w SOLUTION TO

$$
\begin{aligned}
-\operatorname{div}(a\nabla w) &= F && \text{in } \Omega,\\
w &= 0 && \text{on } \Gamma_C\\
w &= 0 && \text{on } \Gamma_I.
\end{aligned}
$$

Increment $u := u - w$. Then

$$
\begin{aligned}
-\operatorname{div}(a\nabla u) &= 0 && \text{in } \Omega,\\
u &= g && \text{on } \Gamma_C\\
a\partial_{\boldsymbol{n}} u &= \varphi - (a\partial_{\boldsymbol{n}} w) && \text{on } \Gamma_C.\\
u &= ? && \text{on } \Gamma_I.
\end{aligned}
$$

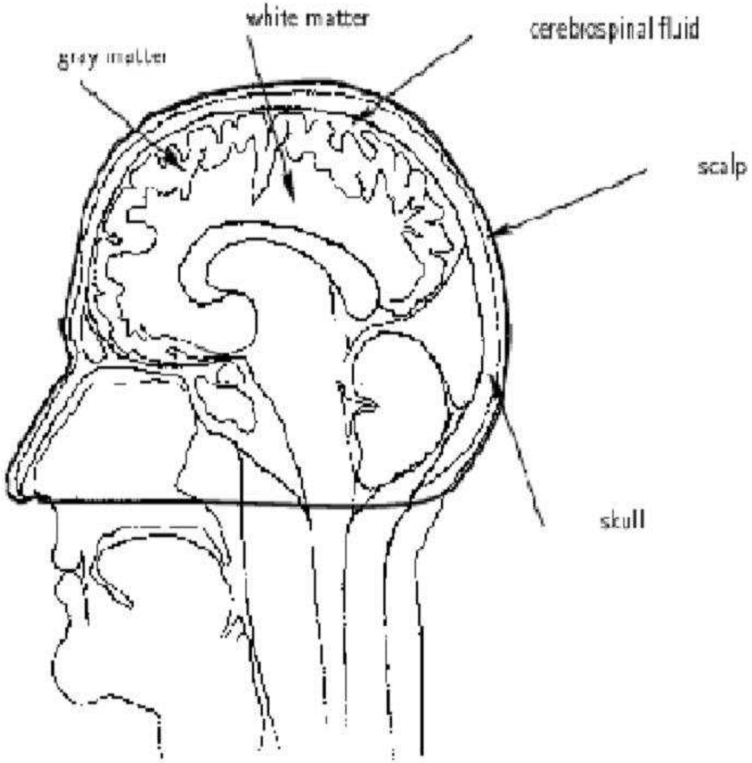
Motivations

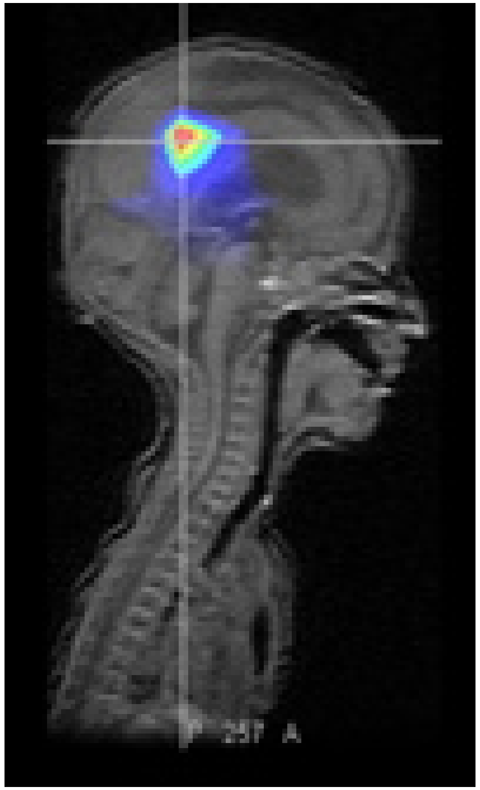
- Data Completion (Many publications! Ben Abda, Bourgeois, . . .)

- Non destructif testing : Detection : Cracks, Contact Resistance, Corrosion Factor , (. . .). —(Ben ABDA, Andrieux, Delvare, . . .)

- Non invasive Diagnosis : Electrical source identification in biology and medical field.
  — EEG, MEG —Epilepsy— (El Badia, Löhrengel, Derbas, . . .),
  — ECG —Ischemia—(Zemzemi, . . .).

EEG (⟹) Electrical activity





Voltage variations recorded on the scalp. (⟹) Transfer the potential on the cortex surface (⟹) Solve Data Completion on the scalp, the skull and the cerebrospinal fluid layer.

~~Well~~ (Ill!) Posedness?

- *Uniqueness?* : Yes! Holmgren Theorem, for smooth solutions. Extended to less regular solutions, by means Carleman inequalities (Modern analysis, many (a lot of!) papers).

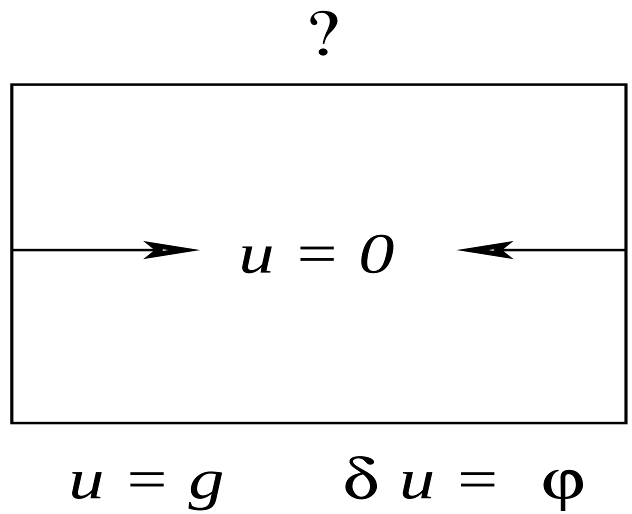
- *Stability?* : No! Tiny deviations on (g, φ) ($\Longrightarrow$) Heavy oscillations on u.
Perturbations are strongly amplified.
Illustrated by J. Hadamard through an example, during the 1920's. The prolem is therefore ill posed.

- *Existence?* : May fail, according to the data (g, φ).

Instability (Hadamard)?

$$g(x) = \sqrt{\frac{2}{\pi}} \sum_{k\geq 1} g_k \sin(kx), \qquad \sum_{k\geq 1} k(g_k)^2 < \infty$$

$$\varphi(x) = \sqrt{\frac{2}{\pi}} \sum_{k\geq 1} \varphi_k \sin(kx), \qquad \sum_{k\geq 1} \frac{1}{k}(\varphi_k)^2 < \infty$$

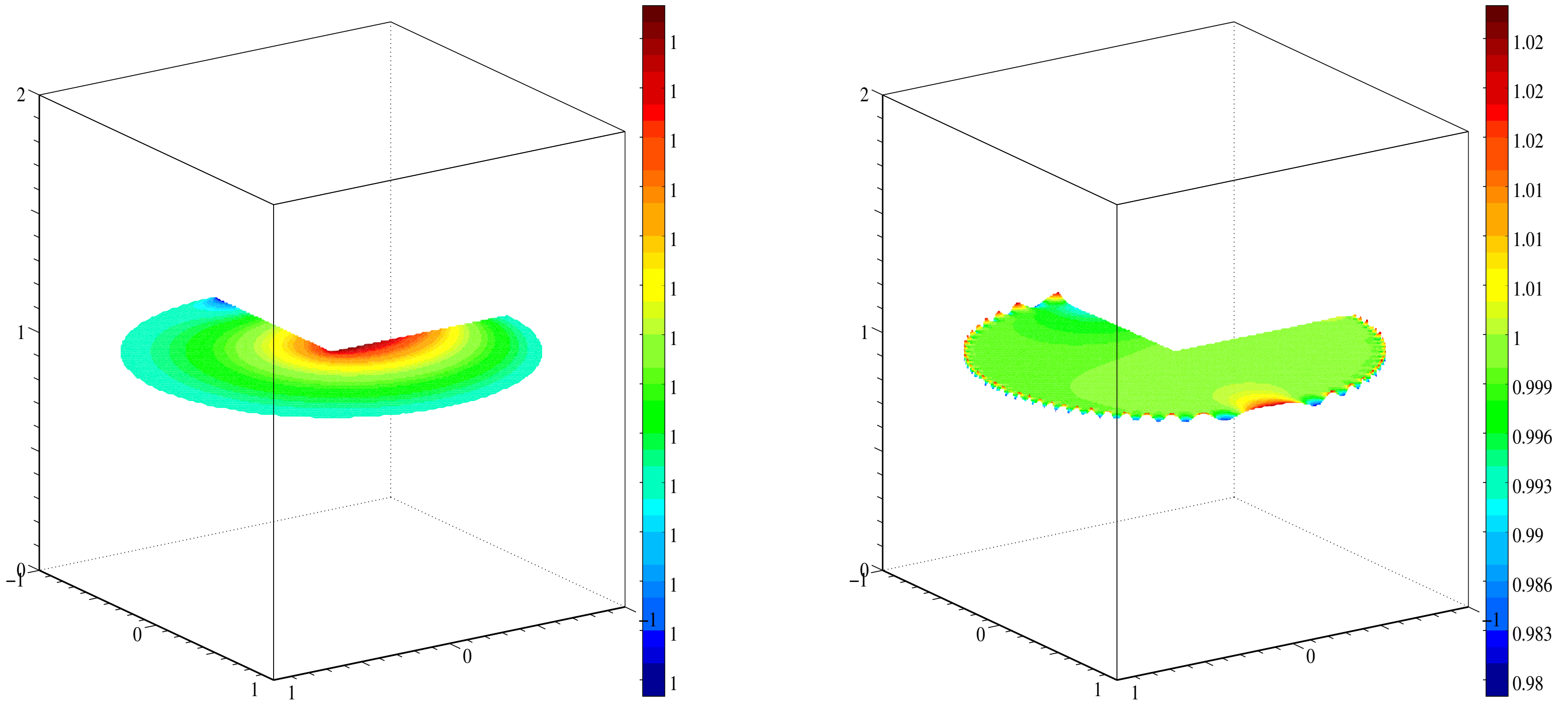


$$u(x,y) = \sqrt{\frac{2}{\pi}} \sum_{k\geq 1} \Big(g_k \cosh(ky) + \frac{\varphi_k}{k} \sinh(ky) \Big) \sin(kx).$$

$$\|u(x,1)\|^2_{L^2(0,\pi)} = \sum_{k\geq 1} \Big(g_k \cosh(k) + \frac{\varphi_k}{k} \sinh(k) \Big)^2 = \infty \ (!)$$

Numerical Example

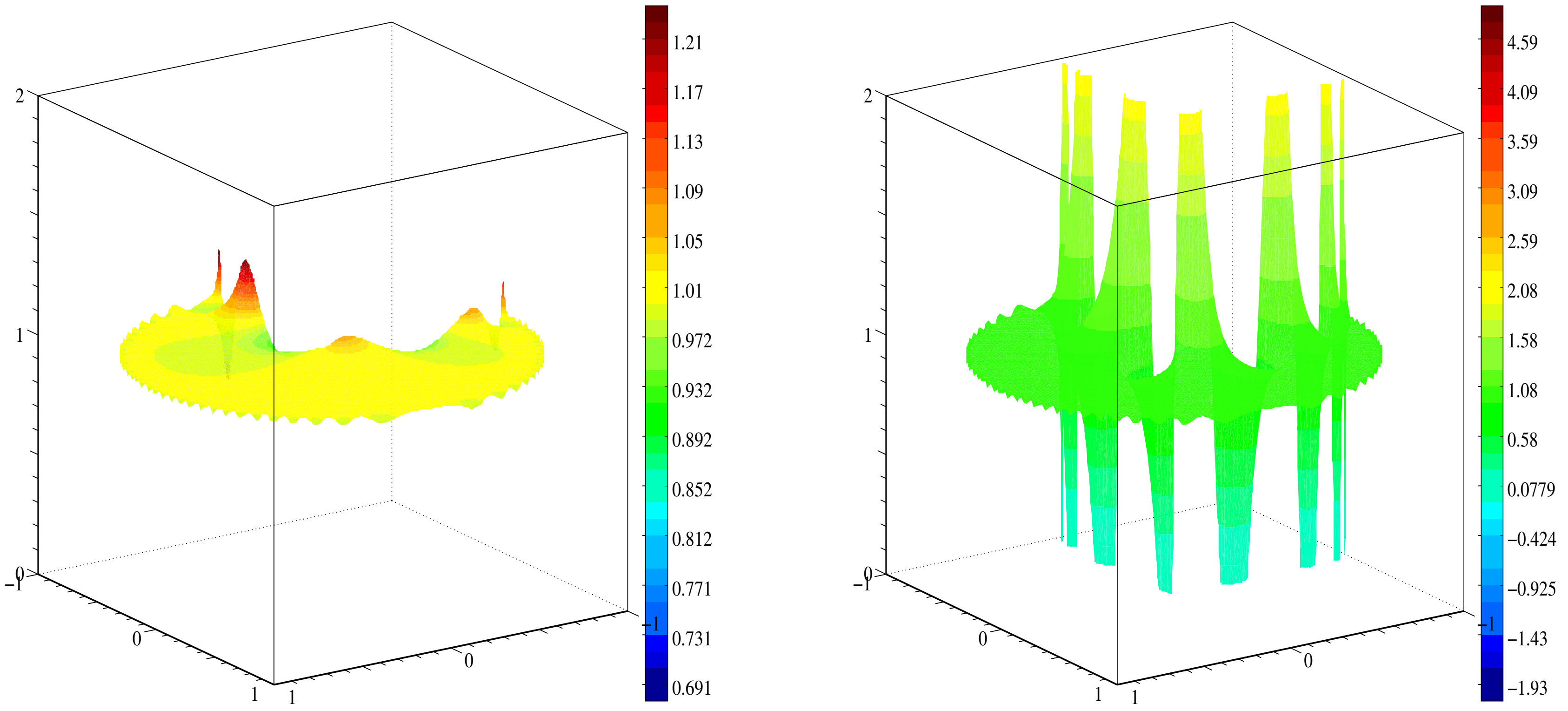
Exact potential $u = 1$.



Γ_C = Circle, Γ_I = Right angle segments

no noise on Dirichlet g (Left) , noise 2% (right) .

Example (continued)



Iterations pursued.

Stop iterations before blow-up (⟹) Regularization.

Existence? The Data?

- *Admissible* data

$$T_{AD} = \Big\{(g,\varphi) = \text{ trace and flux on } \Gamma_C \text{ of a potential with finite energy}\Big\}.$$

Def. 1 *Exact Data? : Those for which solving the data completion problem succeeds!*

Def. 2 *InExact Data? : Those for which solving the data completion problem fails!*

No explicit characterization for neither of them.

The space of ExData is dense in T_{AD}. The set of InExData is dense in T_{AD}.

These tell a lot of things about the resolvent opertor

$$R : \text{ExData} \subset T_{AD} \to V, \qquad (g,\varphi) \mapsto u$$

is a unbounded operator with a dense domain. It is closed (closed graph), injective and has a non-closed range.

Variational Formulation (energetic!)

Admissible space (Trialfunctions space)

$$V_g = \Big\{ v; \quad \nabla v \;\; \text{square integrable}, \qquad v_{|\Gamma_C} = g \Big\}.$$

Testfunctions space

$$W = \Big\{ w; \quad \nabla w \;\; \text{square integrable}, \qquad w_{|\Gamma_I} = 0 \Big\}.$$

Find the potential $u \in V_g$ such that

$$\int_\Omega a \nabla u \nabla w \, d\boldsymbol{x} = \int_{\Gamma_C} \varphi w \, d\Gamma, \qquad \forall w \in W.$$

SOLUTION $\neq$ TEST FUNCTIONS, IN NATURE!

REM. 1 *This formulation may useful on mathematical ground. It is scarcely used in the computations.*

Variational Formulation: BiLaplacian.

Admissible space (Trialfunctions space) $(a(\cdot) \equiv 1)$

$$V_D = \Big\{ v; \quad (\nabla v, Hv) \text{ square integrable}, \qquad (v, \partial_{\boldsymbol{n}} v)_{|\Gamma_C} = (g, \varphi) \Big\}.$$

Testfunctions space

$$W = \Big\{ w; \quad (\nabla w, Hw) \text{ square integrable}, \qquad (w, \partial_{\boldsymbol{n}} w)_{|\Gamma_C} = (0, 0) \Big\}.$$

Find the potential $u \in V_D$ such that

$$\int_\Omega (\Delta u)(\Delta w)\, d\boldsymbol{x} = \int_{\Gamma_C} \varphi w \, d\Gamma, \qquad \forall w \in W.$$

REM. 2 *Formulation successfully used with the Quasi-reversiblity method (see later on)*

Domain Decomposition Approach

Kohn-Vogelius Duplication

Dirichlet-to-Neuman Formulation

Kohn-Vogelius Duplication

Find $u_D \in H^1(\Omega)$ *such that*

$$\begin{cases} -\operatorname{div}(a\nabla u_D) = 0 & \text{in } \Omega, \\ u_D = g & \text{on } \Gamma_C, \end{cases}$$

Find $u_N \in H^1(\Omega)$ *such that*

$$\begin{cases} -\operatorname{div}(a\nabla u_N) = 0 & \text{in } \Omega, \\ a\partial_{\boldsymbol{n}} u_N = \varphi & \text{on } \Gamma_C, \end{cases}$$

Holmgren tells that: If

$$u_D = u_N, \quad a\partial_{\boldsymbol{n}} u_D = a\partial_{\boldsymbol{n}} u_N, \qquad \text{on } \Gamma_I.$$

Then

$$u_D = u_N = u, \text{ the solution to Cauchy problem.}$$

Steklov-Poincaré Method

Take $\mu \in H^{1/2}(\Gamma_I)$. Let $u_D(\mu, g) \in H^1(\Omega)$ *et* $u_N(\mu, \varphi) \in H^1(\Omega)$ *be such that*

$$\left\{ \begin{array}{rcll} -\operatorname{div}\,(a\nabla u_D(\mu, g)) & = & 0 & \text{in } \Omega, \\ u_D(\mu, g) & = & g & \text{sur } \Gamma_C, \\ u_D(\mu, g) & = & \mu & \text{sur } \Gamma_I. \end{array} \right. \qquad \left\{ \begin{array}{rcll} -\operatorname{div}\,(a\nabla u_N(\mu, \varphi)) & = & 0 & \text{in } \Omega, \\ a\partial_{\boldsymbol{n}} u_N(\mu, \varphi) & = & \varphi & \text{on } \Gamma_C, \\ u_N(\mu, \varphi) & = & \mu & \text{on } \Gamma_I. \end{array} \right.$$

Solve equation: *find* $\lambda \in H^{1/2}(\Gamma_I)$ *such that*

$$a\partial_{\boldsymbol{n}} u_D(\lambda, g) - a\partial_{\boldsymbol{n}} u_N(\lambda, \varphi) = 0, \qquad \text{on } \Gamma_I$$

or in a splitted form

$$a\partial_{\boldsymbol{n}} u_D(\lambda) - a\partial_{\boldsymbol{n}} u_N(\lambda) = -a\partial_{\boldsymbol{n}} \breve{u}_D(g) + a\partial_{\boldsymbol{n}} \breve{u}_N(\varphi), \qquad \text{in } H^{-1/2}(\Gamma_I)$$

Variational Problem (I)

Find $\lambda \in H^{1/2}(\Gamma_I)$ *such that*

$$s(\lambda, \mu) \quad = \quad \ell(\mu), \qquad \forall \mu \in H^{1/2}(\Gamma_I).$$

The bilinear form is defined by

$$\begin{aligned} s(\lambda, \mu) &= \int_\Omega a \nabla u_D(\lambda) \nabla u_D(\mu) \, d\boldsymbol{x} - \int_\Omega a \nabla u_N(\lambda) \nabla u_N(\mu) \, d\boldsymbol{x} \\ &= s_D(\lambda, \mu) - s_N(\lambda, \mu) \\ &= \int_\Omega a \nabla (u_D(\lambda) - u_N(\lambda)) \nabla (u_D(\mu) - u_N(\mu)) \, d\boldsymbol{x} \end{aligned}$$

The linear form is given by

$$\begin{aligned} \ell(\mu) &= \langle -a\partial_{\boldsymbol{n}} \breve{u}_D(g) + a\partial_{\boldsymbol{n}} \breve{u}_N(\varphi), \mu \rangle_{1/2, \Gamma_C} \\ &= -\int_\Omega a \nabla \breve{u}_D(g) \nabla u_D(\mu) \, d\boldsymbol{x} - \langle \varphi, u_N(\mu) \rangle_{\frac{1}{2}, \Gamma_C}. \end{aligned}$$

Variational Problem (II)

Find $\lambda \in H^{1/2}(\Gamma_I)$ *such that*

$$s(\lambda, \mu) = \ell(\mu), \qquad \forall \mu \in H^{1/2}(\Gamma_I).$$

$$(\Longleftrightarrow) \qquad (\mathcal{S}\lambda := a\partial_{\boldsymbol{n}}(u_D - u_N)(\lambda) = \mathcal{L}) \qquad \text{in } H^{-1/2}(\Gamma_I)$$

Lem. 3 [☺] $s(\cdot,\cdot)$ *is symmetric and positive definite (not elliptic).* [☺]

Lem. 4 [☹] $\mathcal{S}$ *is compact. Degree of compactness* $= \infty$. [☹]

Lem. 5 [☺] *There holds that*

$$\ell(\mu) \leq \eta \|\mu\|_s. \qquad [☺]$$

PROOF (1) : $u_N(\mu)$ is solution to the Laplace equation with $u_{N|\Gamma_I} = \mu$, we have that

$$\|\sqrt{a}\,\nabla u_N(\mu)\|_{L^2}^2 = \min_{v;\ v_{|\Gamma_I}=\mu} \|\sqrt{a}\,\nabla v\|_{L^2}^2. \tag{1}$$

$u_N(\mu)$ is the only minimizer. Since $u_{D|\Gamma_I} = \mu$, then $u_D(\mu)$ is admissible. Hence

$$\|\sqrt{a}\,\nabla u_N(\mu)\|_{L^2}^2 \le \|\sqrt{a}\,\nabla u_D(\mu)\|_{L^2}^2$$

Therefore $s(\mu,\mu) \ge 0$ forall μ.
Now, assume that $s(\mu,\mu) = 0$, this means that $u_D(\mu)$ is minimizer of (1). Then $u_D(\mu) = u_N(\mu) = w$. We obtain that

$$\begin{aligned} -\operatorname{div}(a\nabla w) &= 0 && \text{in } \Omega,\\ w = 0 \qquad (a\partial_{\boldsymbol{n}})w &= 0 && \text{on } \Gamma_C.\\ (w &= \mu? && \text{on } \Gamma_I) \end{aligned}$$

Holmgren Theorem ($\Longrightarrow$) $w = 0$ ($\Longrightarrow$) $\mu = 0$. Hence $s(\cdot,\cdot)$ is positive definite.

PROOF (2): Set $w = (u_D(\lambda) - u_N(\lambda))$, we have

$$\begin{aligned} -\,\mathrm{div}\,(a\nabla w) &= 0 && \text{in } \Omega, \\ w &= 0 && \text{on } \Gamma_I, \\ w &= -u_N(\lambda) && \text{on } \Gamma_C. \end{aligned}$$

Γ_I regular ($\Longrightarrow$) w is smooth around Γ_I.
The Sobolev regularity of

$$\mathcal{S}\lambda = (a\partial_{\boldsymbol{n}} w)_{|\Gamma_I}$$

is high ($\Longrightarrow$) Compactness of $\mathcal{S}$.

Greater smoothness ($\Longrightarrow$) Higher Degree of Compactness

If Γ_I has corners,
The singularities of w are quantified ($\Longrightarrow$) High Degree of Compactness.

PROOF (3) : Show that the energy function is bounded from below

$$\begin{aligned} J(\mu) &= \frac{1}{2} s(\mu,\mu) - \ell(\mu) \\ &= \frac{1}{2} \|\sqrt{a}\,\nabla(u_D(\mu) - u_N(\mu))\|_{L^2}^2 - \ell(\mu) \\ &= \frac{1}{2} \|\sqrt{a}\,\nabla(u_D(\mu,g) - u_N(\mu,\varphi))\|_{L^2}^2 - \frac{1}{2} \|\sqrt{a}\,\nabla(\breve{u}_D(g) - \breve{u}_N(\varphi))\|_{L^2}^2 . \end{aligned}$$

$$(\Longrightarrow) \qquad \min_{\mu} J(\mu) = -\frac{1}{2} \|\sqrt{a}\,\nabla(\breve{u}_D(g) - \breve{u}_N(\varphi))\|_{L^2}^2$$

Then

$$J(t\mu) + \frac{1}{2} \|\sqrt{a}\,\nabla(\breve{u}_D(g) - \breve{u}_N(\varphi))\|_{L^2}^2 \geq 0, \qquad \forall t \in \mathbb{R}.$$

Finir au tableau !

Regularization

Cauchy's problem

The problem I need to solve ($\Longrightarrow$) *Find* $\lambda \in H^{1/2}(\Gamma_I)$ *such that*

$$s(\lambda, \mu) = \ell(\mu), \qquad \forall \mu \in H^{1/2}(\Gamma_I).$$

Exact data (g, φ) and therefore $\ell(\cdot)$.

Lavrentiev's problem

Find $\lambda_\alpha \in H^{1/2}(\Gamma_I)$ *such that*

$$\alpha s_D(\lambda_\alpha, \mu) + s(\lambda_\alpha, \mu) = \ell(\mu), \qquad \forall \mu \in H^{1/2}(\Gamma_I).$$

Convergence

Does $(\lambda_\alpha)_\alpha$ converge towards λ?
With respect to the weak norm $\|\cdot\|_s = \sqrt{s(\cdot,\cdot)}$?
With respect to the strong norm $\|\cdot\|_{s_D} = \sqrt{s_D(\cdot,\cdot)}$?

LEM. **6** *We have*

$$\|\lambda_\alpha - \lambda\|_s \leq \frac{\sqrt{\alpha}}{2}\|\lambda\|_{s_D}.$$

and

$$\lim_{\alpha\to 0} \|\lambda_\alpha - \lambda\|_{s_D} = 0.$$

PROOF : $s(\cdot,\cdot)$ (or $\mathcal{S}$) is symmetric and compact. Diagonalize it with respect to $s_D(\cdot,\cdot)$. The eigenvalues $((\gamma_m)_m > 0)$ decay towards 0. Then,

$$\lambda(\boldsymbol{x}) = \sum_{m\geq 0} \frac{\ell_m}{\gamma_m} e_m(\boldsymbol{x}) = \sum_{m\geq 0} \lambda_m e_m(\boldsymbol{x})$$

Easy computations produces

$$\|\lambda_\alpha - \lambda\|_{s_D}^2 = \sum_{m\geq 0} \left[\frac{\alpha}{\alpha+\gamma_m}\right]^2 (\lambda_m)^2 \leq \sum_{m\geq 0} (\lambda_m)^2 \leq \|\lambda\|_{s_D}^2.$$

Use Lebesgue's Dominated convergence theorem. Moreover

$$\|\lambda_\alpha - \lambda\|_s^2 = \alpha \sum_{m\geq 0} \left[\frac{\sqrt{\alpha\gamma_m}}{\alpha+\gamma_m}\right]^2 (\lambda_m)^2 \leq \frac{\alpha}{4} \sum_{m\geq 0} (\lambda_m)^2 \leq \frac{\alpha}{4} \|\lambda\|_{s_D}^2.$$

Noisy data

Measurements are inaccurate ($\Longrightarrow$) Noisy Data $(g + \delta g, \varphi + \delta\varphi)$.

Noise intensity ϵ

$$\|\delta g\|_{H^{1/2}(\Gamma_I)} \leq \epsilon, \qquad \|\delta\varphi\|_{H^{-1/2}(\Gamma_I)} \leq \epsilon$$

Regularized solution $\lambda_\epsilon (=: \lambda_{\epsilon,\alpha}) \in H^{1/2}(\Gamma_I)$ *telle que*

$$\alpha s_D(\lambda_\epsilon, \mu) + s(\lambda_\epsilon, \mu) = (\ell + \delta\ell)(\mu), \qquad \forall \mu \in H^{1/2}(\Gamma_I).$$

Chose α, how?
Be consistent (accuracy).
Control the computations (stability).

Convergence

THÉO. **7** *The following holds*

$$\|\lambda_\epsilon - \lambda\|_{s_D} \leq e_\alpha + \frac{\epsilon}{\sqrt{\alpha}}.$$

with $e_\alpha \to 0$ *when* $\alpha \to 0$.

Assume that $\alpha = \alpha(\epsilon)$ *is chosen such that*

$$\lim_{\epsilon \to 0} \alpha(\epsilon) = 0, \qquad \lim_{\epsilon \to 0} \frac{\epsilon}{\sqrt{\alpha}} = 0.$$

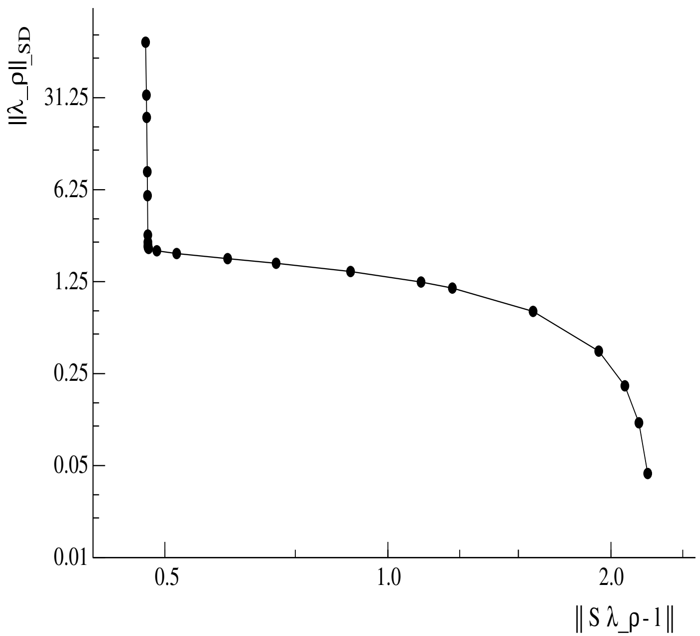
Then we have

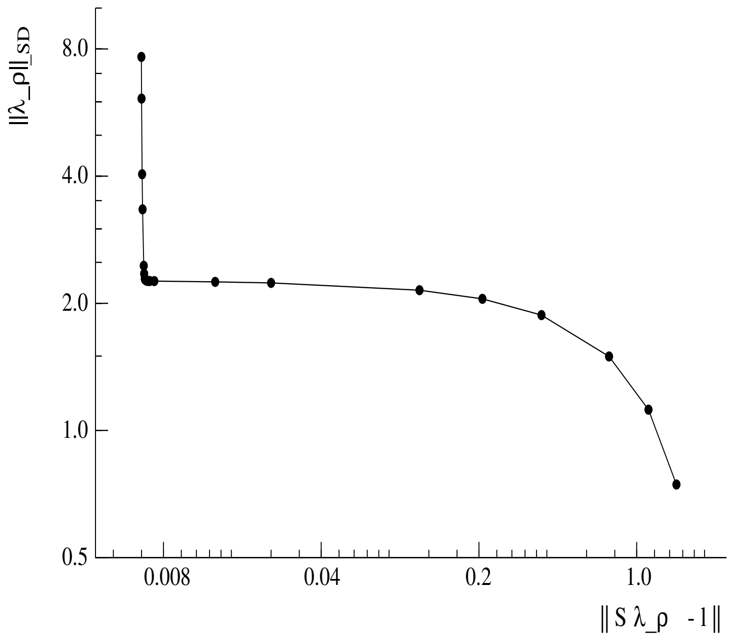
$$\lim_{\epsilon \to 0} \|\lambda_\epsilon - \lambda\|_{s_D} = 0.$$

The parameter α?

Plot the L-curve $\qquad \alpha \mapsto (\|\lambda_\alpha\|, \|\mathcal{S}\lambda_\alpha - \mathcal{L}\|)$.

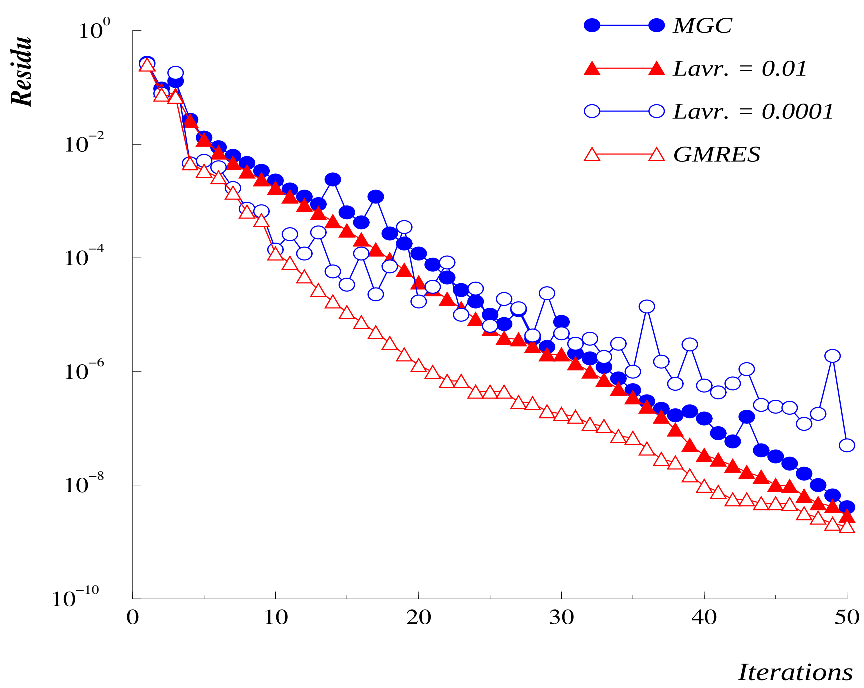
Pick-up the α of the corner.

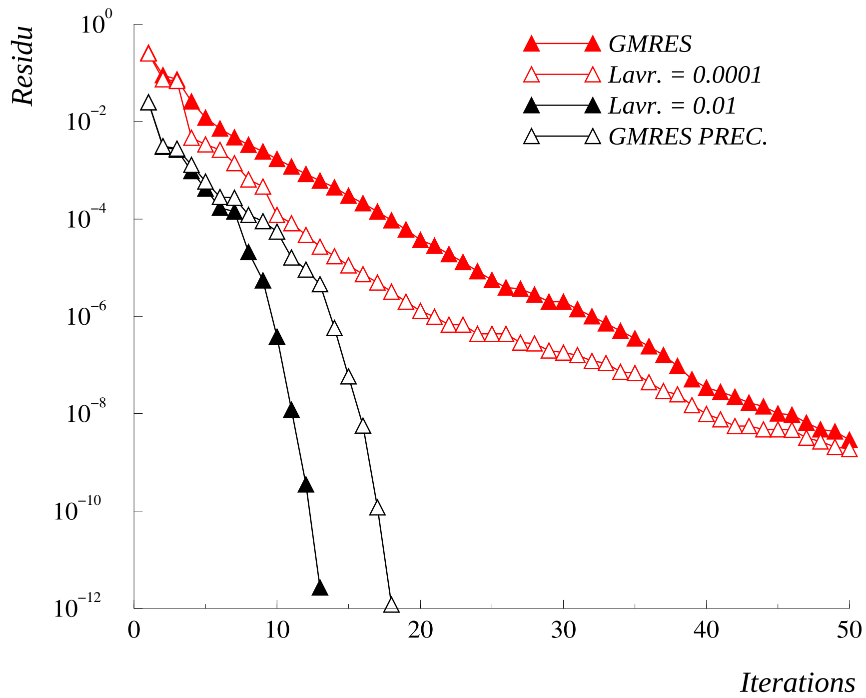




L-curve to choose α.

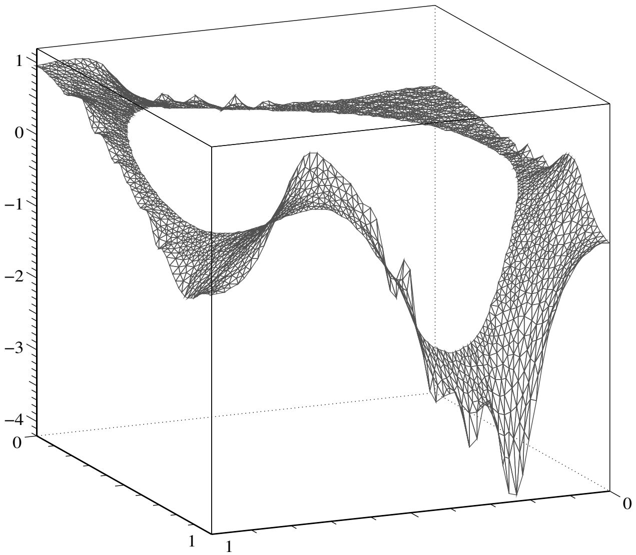
(left) Noisy Dirichlet data. (right) Noisy Neumann data.

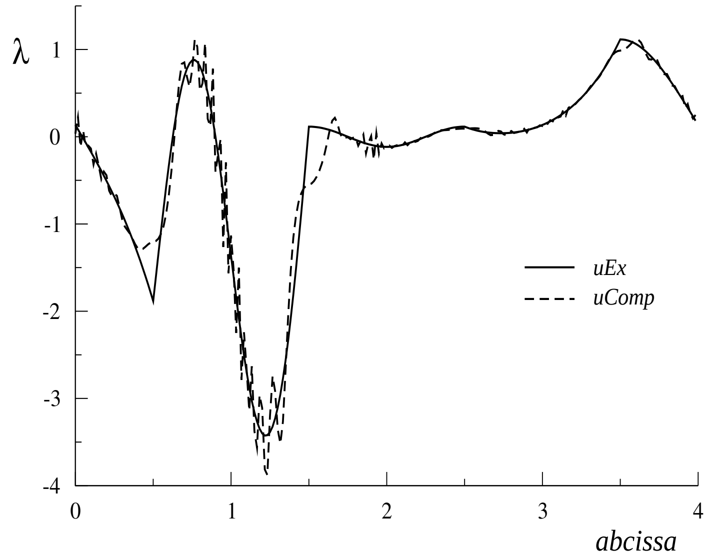




Conjugate Gradient and GMRES (left),
Peconditoned GMRES et GMRES (right).
Précond. $(\mathcal{S}_D)^{-1}(\alpha\mathcal{S}_D + \mathcal{S})\lambda_\alpha = \ell$,

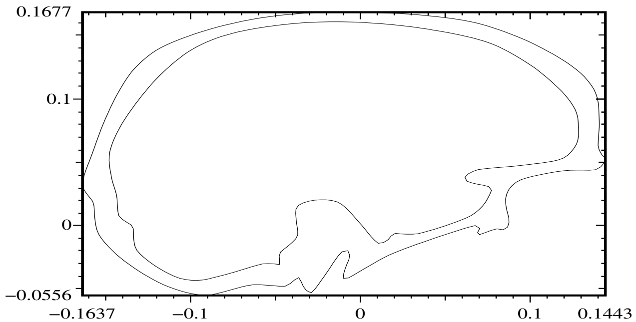
Example I

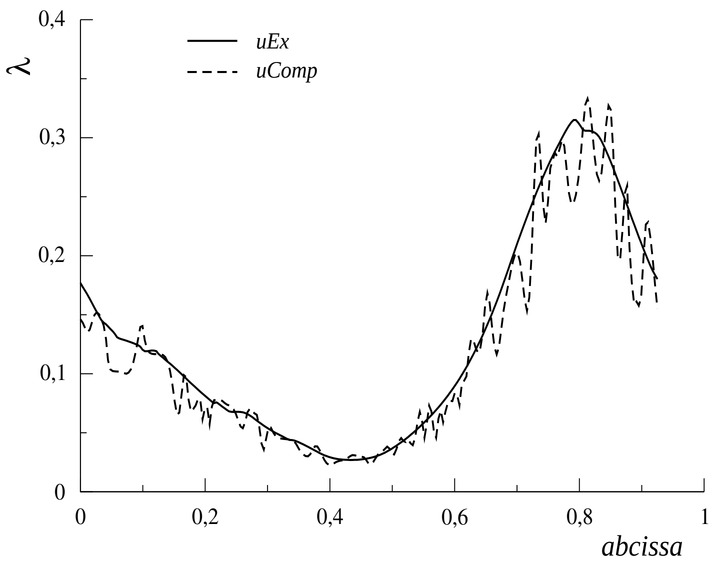




The computed solution. Exact versus computed u_N.

Example II





The computed solution. Exact versus computed u_N.

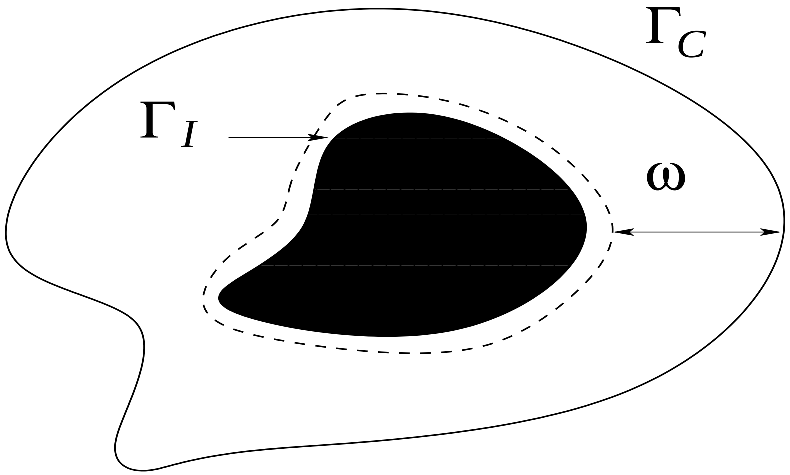
Local SuperConvergence

Exact Data

$|u - u_N(\lambda_\alpha, \varphi)|_{a,H^1(\Omega)} = |u_N(\lambda - \lambda_\alpha)|_{a,H^1(\Omega)} \leq C$

A localized estimate at vicinity of Γ_C,

$|u_N(\lambda - \lambda_\alpha)|_{H^{1/2}(\Gamma_C)} \leq \|\lambda - \lambda_\alpha\|_s \leq C\sqrt{\alpha}\|\lambda\|_{s_D}.$



What about intermediary domains, such as ω?

An issue of *'Geometrical Interpolation'* ($\Longrightarrow$) Carleman Estimates (Tataru).

Bias

PROP. 8 *There exists* $q = q(\omega) \in]0, 1/2[$ *such that*

$$|u - u_N(\lambda_\alpha, \varphi)|_{a,H^1(\omega)} \leq C\alpha^q \|\lambda\|_{s_D}.$$

Bias-Variance

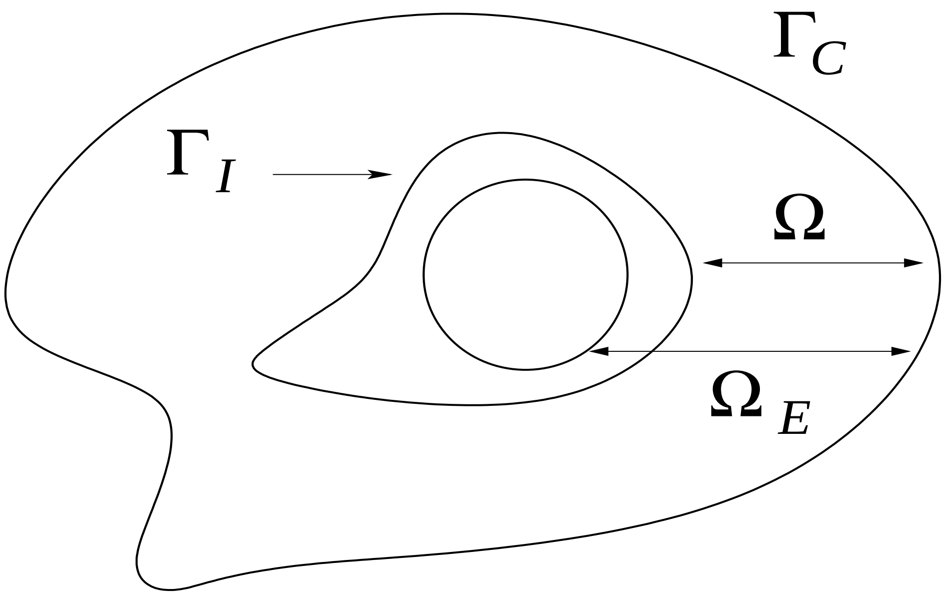
PROP. 9

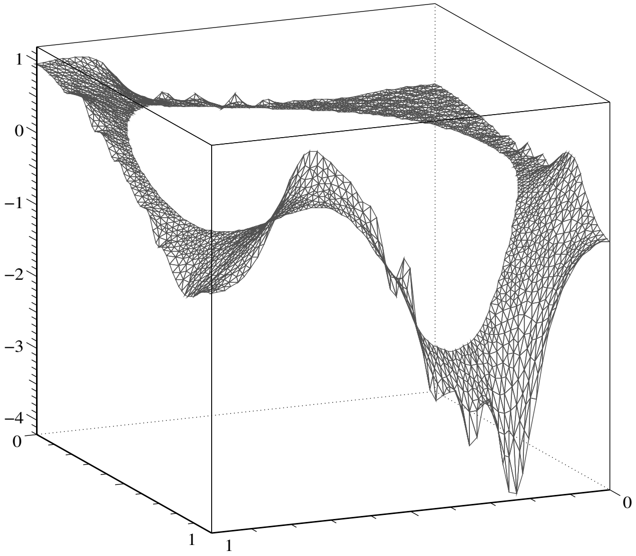
$$|u - u_N(\lambda_\epsilon, \varphi_\epsilon)|_{H^1(\omega)} \leq C\alpha^q \left(\|\lambda\|_{s_D} + \frac{\epsilon}{\sqrt{\alpha}}\right).$$

Extended-Domain Method

Proceed as follows

1. Extend the domain in the Incomplete boundary side.
2. Carry out computations in the extended domain,
3. Restrict to the real domain.

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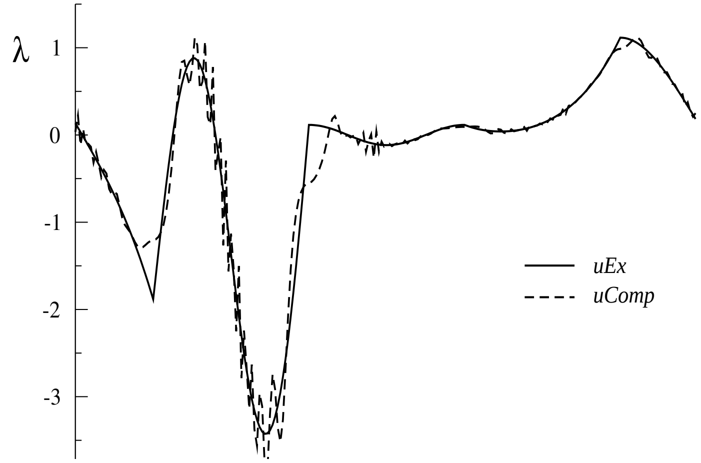
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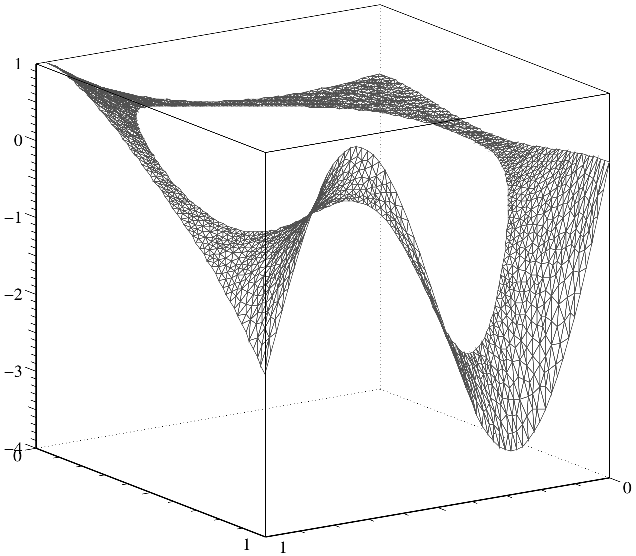
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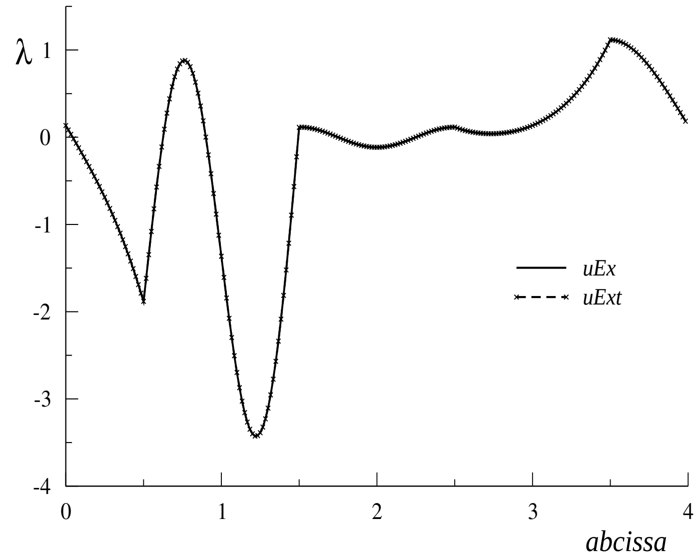
λ
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abcissa
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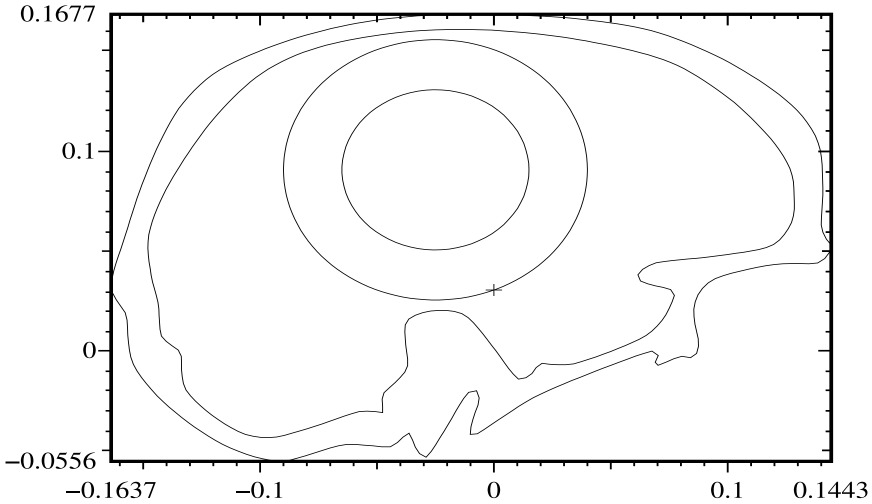
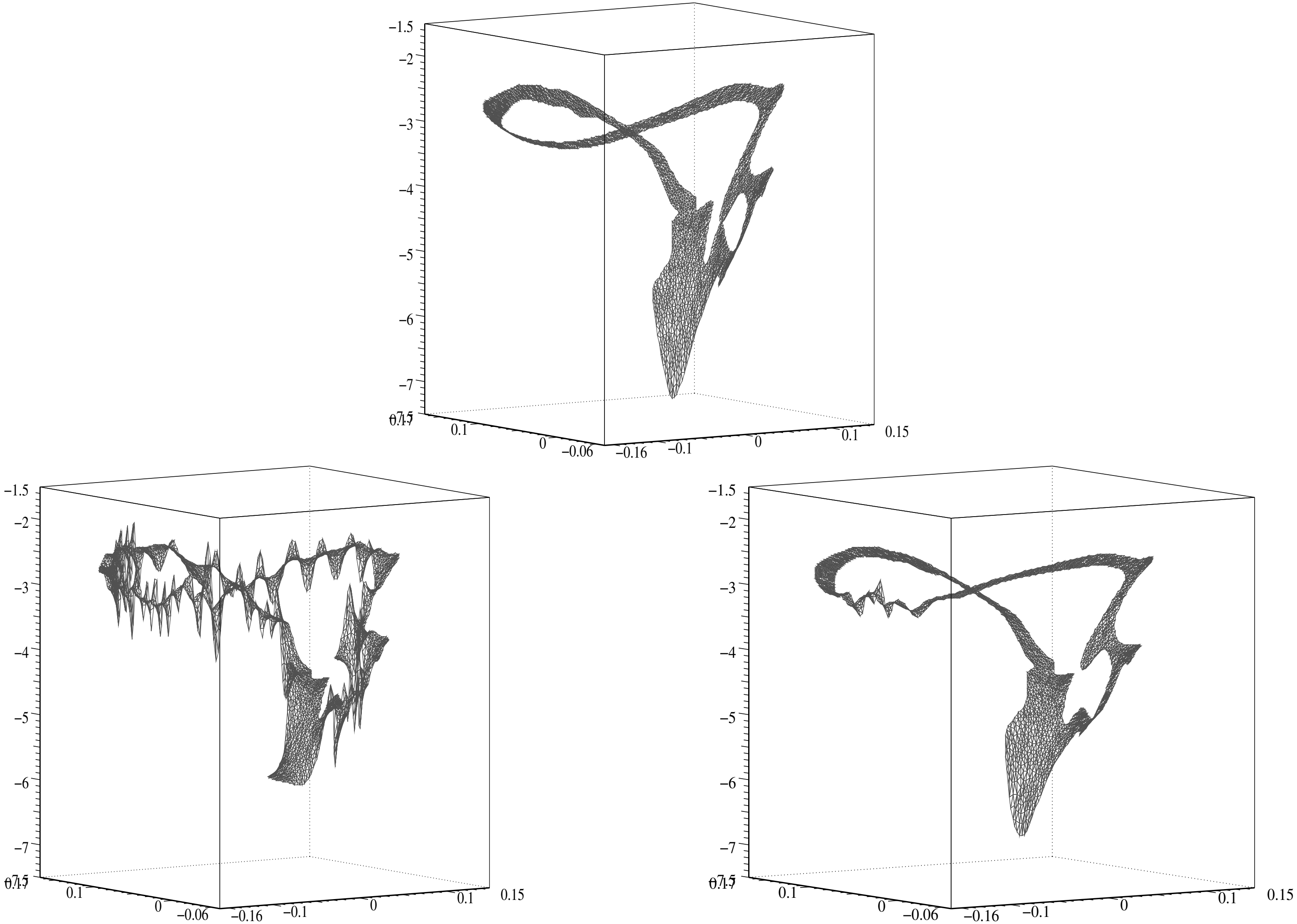
Brain like layer



(External Γ_C , Internal Γ_I))
Circles, Extended boundary

The electric potential is created by
a mono-polar or dipolar source located at the crosspoint.



−1.5
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−5
−6
−7
−7.5
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0.1
0
−0.06
−0.16
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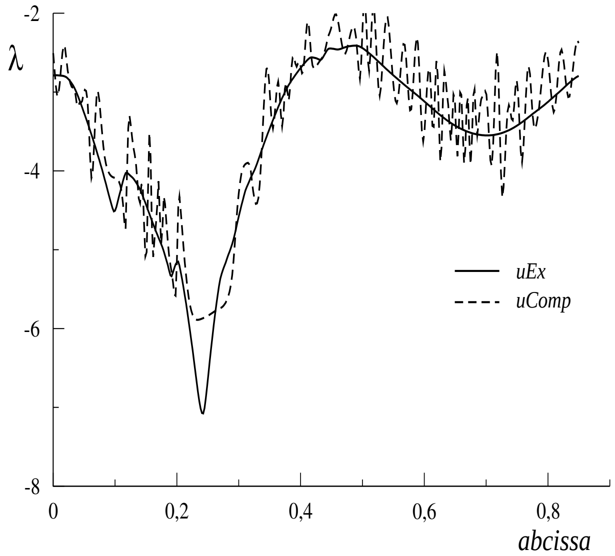
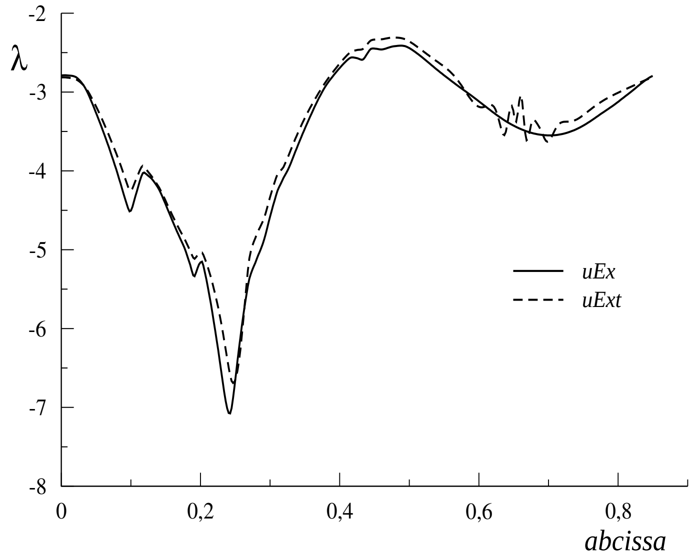
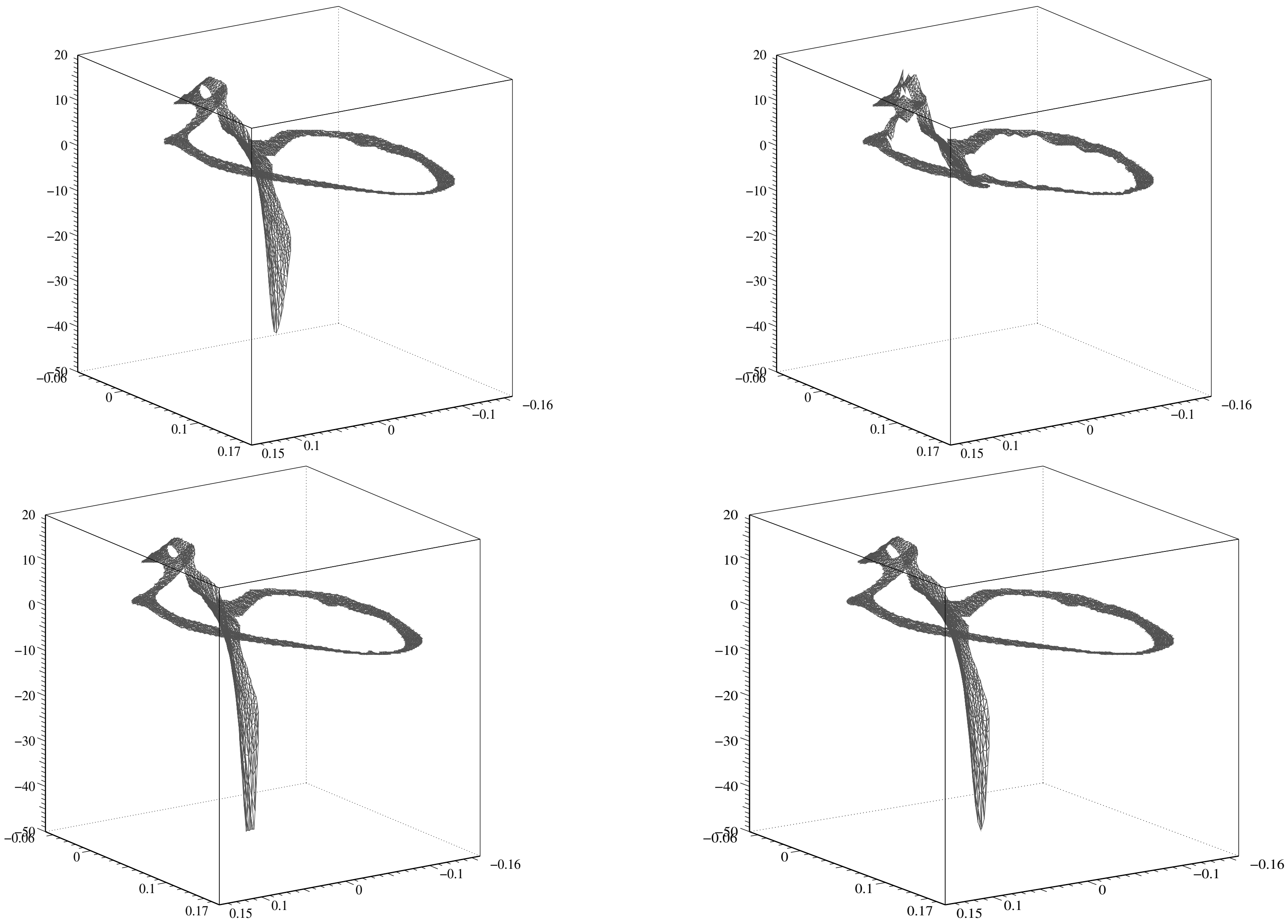
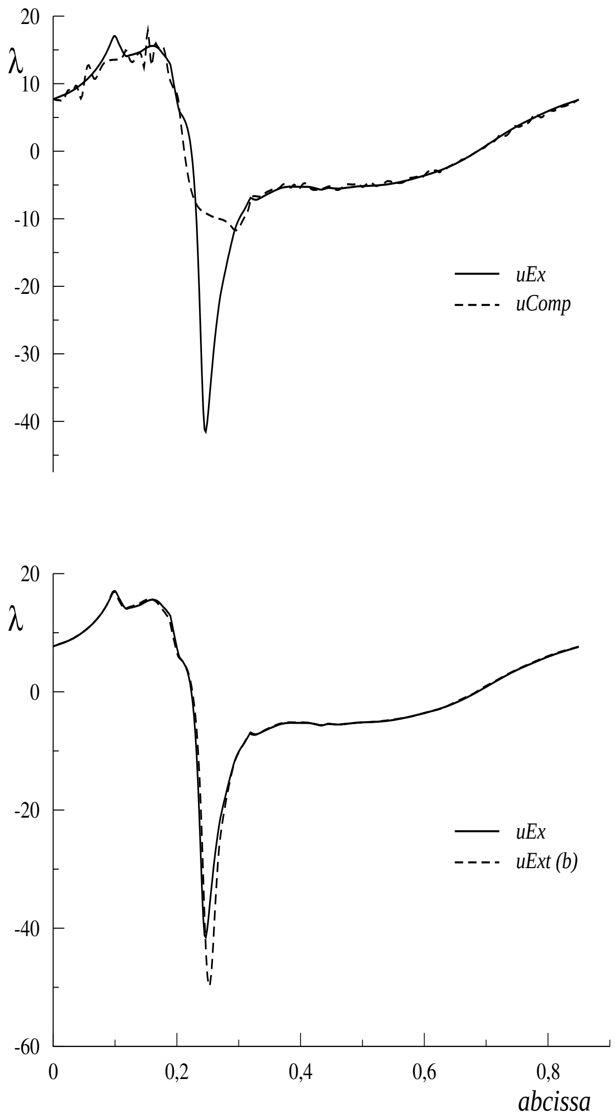
0
0.1
0.15


λ
-2
-4
-6
-8
0
0,2
0,4
0,6
0,8
abcissa
uEx
uComp

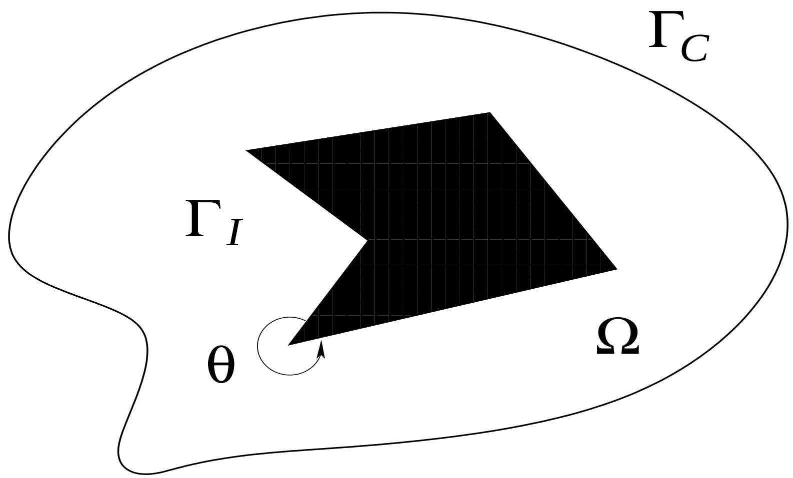

λ
-2
-3
-4
-5
-6
-7
-8
0
0,2
0,4
0,6
0,8
abcissa
uEx
uExt


20
10
0
−10
−20
−30
−40
−50
−0.06
0
0.1
0.17
0.15
0.1
0
−0.1
−0.16



Noise 10%. Automatic selection of α except for the last case (optimal).

Finite Element Method



Restrict FEM to Γ_I.

$\mathcal{T}_h$, : a regular triangulation of Γ_I, polygonal or polyhedral.

The finite element space in Γ_I

$$H_h = \Big\{ \psi_h \in \mathscr{C}(\Gamma_I), \qquad \forall \tau \in \mathcal{T}_h, \quad \psi_{h|\tau} \in \mathcal{P}_1 \Big\}.$$

Semi-Discrete Problem

The (semi)-discrete problem amounts to : *find* $\lambda_{\alpha,h} \in H_h$ *such that*

$$\alpha s_D(\lambda_{\alpha,h}, \mu_h) + s(\lambda_{\alpha,h}, \mu_h) = \ell(\mu_h), \qquad \forall \mu_h \in H_h.$$

$(\Longleftrightarrow)$ *Find* $\lambda_{\alpha,h}$ *such that*

$$\int_\Omega a\left((1+\alpha)\nabla u_D(\lambda_{\alpha,h})\nabla u_D(\mu_h)\, d\boldsymbol{x} - \nabla u_N(\lambda_{\alpha,h})\nabla u_N(\mu_h)\right)\, d\boldsymbol{x} =$$

$$-\int_\Omega a\nabla\breve{u}_D(g)\nabla u_D(\mu_h)\, d\boldsymbol{x} - \langle\varphi, u_N(\mu_h)\rangle_{\frac{1}{2},\Gamma_C},$$

$$\forall \mu.$$

Bias ($\Longrightarrow$) Exact Data

PROP. **10** *Let* $p < p^*(< 1/2)$. *We have*

$$\|\lambda_\alpha - \lambda_{\alpha,h}\|_{s_D} \le C \left(e_\alpha + e_h + \sqrt{\frac{h^{1+2p}}{\alpha}} \right) \|\lambda\|_{s_D}.$$

$$\|\lambda_\alpha - \lambda_{\alpha,h}\|_s \le C \left(\sqrt{\alpha} + h^{1+2p} \right) \|\lambda\|_{s_D}.$$

Bias-Variance

Noisy Data $(g + \delta g, \varphi + \delta\varphi)$ with

$$\|\delta g\|_{H^{1/2}(\Gamma_I)} \leq \epsilon, \qquad \|\delta\varphi\|_{H^{-1/2}(\Gamma_I)} \leq \epsilon$$

THÉO. **11** *Let* $p < p^*(< 1/2)$. *We have*

$$\|\lambda - \lambda_{\alpha,h}\|_{s_D} \leq C\left(e_\alpha + e_h + \sqrt{\frac{h^{1+2p}}{\alpha}}\right)\|\lambda\|_{s_D} + \frac{\epsilon}{\sqrt{\alpha}},$$

and

$$\|\lambda - \lambda_{\alpha,h}\|_s \leq C\left(\sqrt{\alpha} + h^{1/2+p}\right)\|\lambda\|_{s_D} + \epsilon.$$

REM. 3 *The convergence is driven by* ϵ. *Chosee* $\alpha = \alpha(\epsilon)$ *and* $h = h(\epsilon)$ *s. t.*

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\sqrt{\alpha}} = 0, \qquad \lim_{\epsilon\to 0}\sqrt{\frac{h^{1+2p}}{\alpha}} = 0.$$

Full-Discretization?

The full-discrete problem amounts to : *find* $\lambda_{\alpha,h} \in H_h$ *such that*

$$\alpha s_{D,h}(\lambda_{\alpha,h}, \mu_h) + s_h(\lambda_{\alpha,h}, \mu_h) = \ell(\mu_h), \qquad \forall \mu_h \in H_h.$$

$(\Longleftrightarrow)$ *Find* $\lambda_{\alpha,h}$ *such that*

$$\int_\Omega a\Big((1+\alpha)\nabla u_{D,h}(\lambda_{\alpha,h})\nabla u_{D,h}(\mu_h) - \nabla u_{N,h}(\lambda_{\alpha,h})\nabla u_{N,h}(\mu_h)\Big)\,d\boldsymbol{x} = - \int_\Omega a\nabla \breve{u}_{D,h}(g)\nabla u_{D,h}(\mu_h)\,d\boldsymbol{x} - \langle \varphi, u_{N,h}(\mu_h)\rangle_{\frac{1}{2},\Gamma_C}, \qquad \forall \mu_h.$$

Existence? Uniqueness

	Existence	Uniqueness
Continuous	NO!	YES!
Semi-discrtete	YES!	YES!
Full-discrtete	YES!	NO!

Table 1: Cauchy's problem without regularization.

Conclusion

DATA COMPLETION PROBLEM, APPARANTLY EASY!

`Mathematical Analysis:` SUBSTANTIAL PROGRESS TO UNDERSTAND WHAT HAPPENS.

`Computations` : HARD, FAR FROM DESIRED OBJECTIVES, ... AT LEAST IN 3D.

Extension

The full Discrete Varitarional Problem

($\Longleftrightarrow$) Change $u_D(\lambda_h)$ and $u_N(\lambda_h)$ into $u_{D,h}(\lambda_h)$ and $u_{N,h}(\lambda_h)$

ALMOST DONE!

That's all!

Thank you!